Rolling Without Slipping

Consider the wheel shown below that rolls along a rough, stationary horizontal surface with the center of the wheel O having a velocity and acceleration of v_O and a_O, respectively. As the wheel moves, it is assumed that sufficient friction acts between the wheel and the fixed horizontal surface that the contact point C does not slip. The consequences of the "no slip" condition at C are:

$$v_{cx} = 0$$

C cannot go into the ground. $\Rightarrow V_{cy} = 0$

$$a_{cx} = 0$$

We can show that $\vec{V}_C = 0 \Rightarrow a_{Cx} = 0, \ a_{Cy} = \omega^2 R \hat{j}$

(If either of the above is not true, then point C "slips" as the wheel moves.)

at P, $\vec{V}_p = \vec{V}_o + \vec{\omega} \times (R\cos\theta\,\hat{i} - R\sin\theta\,\hat{j})$

$= (\omega R - \omega R\sin\theta)\hat{i} - \omega R\cos\theta\,\hat{j}$

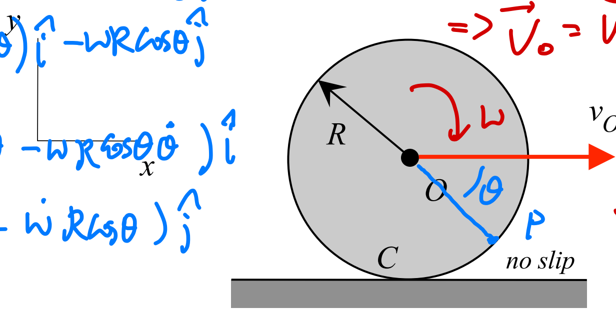
$\vec{a}_p = \dot{\vec{V}}_p = (\dot{\omega} R - \dot{\omega} R\sin\theta - \omega R\cos\theta\,\dot{\theta})\hat{i}$

$+ (\omega R\sin\theta\,\dot{\theta} - \dot{\omega} R\cos\theta)\hat{j}$

$\theta = \frac{\pi}{2}$, i.e., at C $\longrightarrow \begin{cases} \vec{V}_C = 0 \\ \vec{a}_C = \omega^2 R\,\hat{j} \end{cases}$

$\vec{V}_C = 0$

$\Rightarrow \vec{V}_o = \vec{V}_C + \vec{\omega} \times R\hat{j} = \boxed{\omega R\,\hat{i}}$



$\vec{\omega} = -\omega\hat{k}$

$\dot{\theta} = \omega$

We will encounter many problems throughout the course that involve the rolling without slipping of a body on a stationary surface. Although this concept is defined through a simple set of equations, the consequences of rolling without slipping on the velocity and acceleration of other points on the body can become quite complicated. We will see this through a number of examples.

CHALLENGE QUESTION: If C is a no-slip point, what are the y-components for the velocity and acceleration of C?

$V_{oy} = 0, \ a_{oy} = 0$

ANSWER: Since O moves on a straight, horizontal path, the y-components of the velocity and acceleration of O are zero. From this and the no-slip condition above for C, we can write:

$\vec{r}_{C/O} = -R\hat{j}$

$V_{oy} = 0 \Rightarrow$

$$\vec{v}_C = \vec{v}_O + \vec{\omega} \times \vec{r}_{C/O}$$
$$v_{Cy}\hat{j} = v_O\hat{i} + (\omega\hat{k}) \times (-R\hat{j}) = (v_O + R\omega)\hat{i} \qquad \Rightarrow \qquad v_{Cy} = 0$$

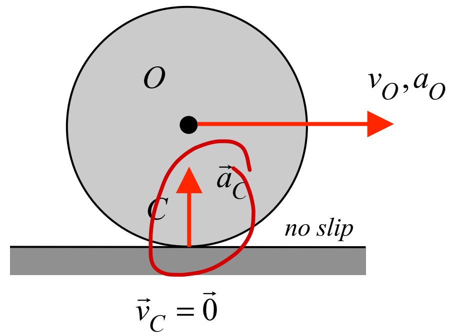
and

$a_{oy} = 0 \Rightarrow$

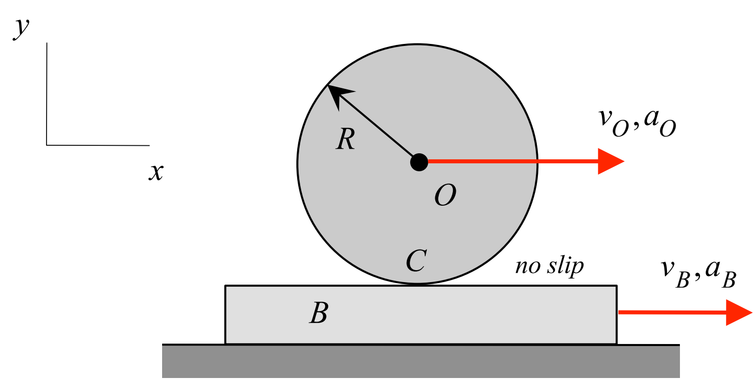
$$\vec{a}_C = \vec{a}_O + \vec{\alpha} \times \vec{r}_{C/O} - \omega^2 \vec{r}_{C/O}$$
$$a_{Cy}\hat{j} = a_O\hat{i} + (\alpha\hat{k}) \times (-R\hat{j}) - \omega^2(-R\hat{j})$$
$$= (a_O + R\alpha)\hat{i} + R\omega^2\hat{j} \qquad \Rightarrow \qquad a_{Cy} = R\omega^2$$

From this we see that: $\vec{v}_c = \vec{0}$ and $\vec{a}_C = R\omega^2\hat{j} \neq \vec{0}$.

This says that the velocity for a no-slip point is zero; however, the acceleration of that point is NOT zero.



Consider another situation where here the wheel rolls without slipping on a second body B that is itself translating in the x-direction.



$\vec{V}_C = V_B \hat{i}$

$\vec{V}_O = \vec{V}_C + (-\omega\hat{k}) \times (R\hat{j})$

$= (V_B + \omega R)\hat{i}$

No slip for this situation is described by the following:

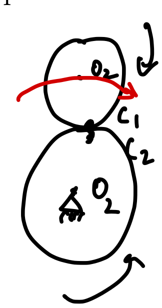
$$v_{cx} = v_B$$

$$a_{cx} = a_B$$

Following the same type of analysis shown above, we can show that the no-slip conditions produces: $v_{Cy} = 0$ and $a_{Cy} \neq 0$.

For all Rolling without slipping problems, we always start with

$V_{Cx} = V_B$, $V_{Cy} = 0$, $a_{Cx} = a_B$



$V_{C_1 x} = V_{C_2 x}$

$V_{C_1 y} = V_{C_2 y} = 0$

$a_{C_1 x} = a_{C_2 x}$

$\boxed{a_{C_1 y} \neq a_{C_2 y}}$

Example 2.A.4

Given: Rack A moves to the right with a speed of v_A and an acceleration of a_A. Rack B moves to the left with a constant speed of v_B. Assume $v_A = 0.8$ m/s, $a_A = 2$ m/s^2, $v_B = 0.6$ m/s, $r = 0.1$ m and $R = 0.16$ m.

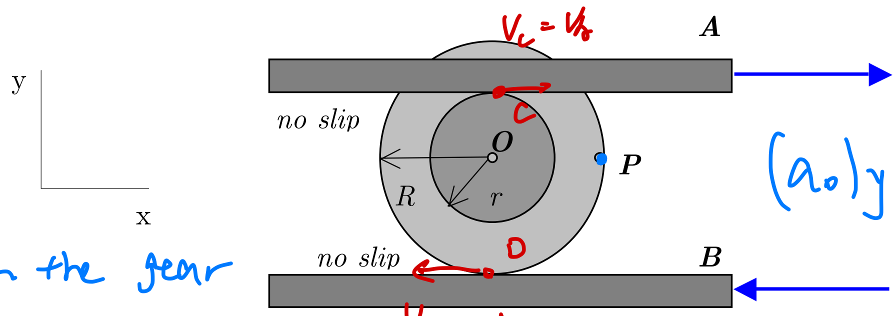
$a_B = 0$

Find: Determine:

(a) The velocity of point P on the outer rim of the gear; and
(b) The acceleration of point P on the outer rim of the gear.

$a_{Cx} = a_A$

$V_C = V_A$



$(a_O)_y = 0$

$V_D = V_B$

$a_{Dx} = a_{Bx} = 0$

Need to find ω, α for the gear

$\vec{v}_C = \vec{V}_D + \vec{\omega} \times \vec{r}_{C/D}$

$= -V_B \hat{i} + \omega \hat{k} \times (R+r)\hat{j}$

$= -[V_B + \omega(R+r)]\hat{i} = V_A \hat{i} \Rightarrow \omega = -\dfrac{V_A + V_B}{R+r}$

$\vec{a}_C = \vec{a}_D + \vec{\alpha} \times \vec{r}_{C/D} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{C/D})$

$= 0\hat{i} + a_{Dy}\hat{j} + \alpha \hat{k} (R+r)\hat{j} + \omega\hat{k} \times [\omega\hat{k} \times (R+r)\hat{j}]$

$= a_{Dy}\hat{j} - \alpha(R+r)\hat{i} - \omega^2(R+r)\hat{j}$

$a_{Cx} = -\alpha(R+r) = a_A \Rightarrow \alpha = \dfrac{-a_A}{R+r}$

find $\vec{V}_P$, $\vec{a}_P$

$\vec{V}_P = \vec{V}_D + \vec{\omega} \times \vec{r}_{P/D} = -V_B\hat{i} + \omega\hat{k} \times (R\hat{i} + R\hat{j})$

$= -(V_B + \omega R)\hat{i} + \omega R\hat{j}$

$\vec{a}_P = \vec{a}_D + \vec{\alpha} \times \vec{r}_{P/D} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{P/D}) = a_{Dy}\hat{j} + \alpha\hat{k} \times (R\hat{i} + R\hat{j}) + \omega\hat{k} \times [\omega\hat{k} \times (R\hat{i} + R\hat{j})]$

$= (-\alpha R - \omega^2 R)\hat{i} + (a_{Dy} + \alpha R - \omega^2 R)\hat{j}$

don't know ← $(a_O)_y = 0$

$$\vec{a}_O = \vec{a}_D + \vec{\alpha} \times \vec{r}_{O/D} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{O/D})$$

$$= a_{Dy}\hat{j} + \alpha\hat{k} \times R\hat{j} + \omega\hat{k} \times (\omega\hat{k} \times R\hat{j})$$

$$= a_{Dy}\hat{j} - \alpha R\hat{i} - \omega^2 R\hat{j}$$

$$a_{Oy} = 0 \Rightarrow a_{Dy} - \omega^2 R = 0 \Rightarrow a_{Dy} = \omega^2 R$$

Substitute to $\vec{a}_P$:

$$\vec{a}_P = (-\alpha R - \omega^2 R)\hat{i} + (\omega^2 R + \alpha R - \omega^2 R)\hat{j}$$

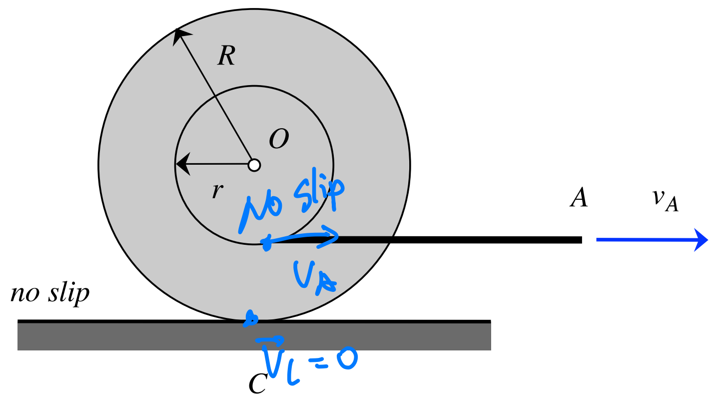
$$= (-\alpha R - \omega^2 R)\hat{i} + \alpha R\hat{j}$$

Example 2.A.6

Given: A cable is wrapped around the inner radius of a spool. End A of the cable is moving to the right with a speed of v_A. The spool is able to roll without slipping on a rough horizontal surface.

Find: Determine:
- (a) The velocity of the center O of the spool; and
- (b) The angular velocity of the spool.

Use the following parameters in your analysis: $v_A = 2$ m/s, $r = 0.4$ m and $R = 0.8$ m. Also, be sure to express your answers as vectors.



$$\vec{V}_A = \vec{V}_C + \vec{\omega} \times \vec{r}_{A/C}$$

$$= \vec{0} + \omega\hat{k} \times (R-r)\hat{j}$$

$$= -\omega(R-r)\hat{i} \qquad = V_A\hat{i}$$

$$\Rightarrow \quad \omega = -\frac{V_A}{R-r}$$

$$\vec{V}_O = \vec{V}_C + \vec{\omega} \times \vec{r}_{O/C}$$

$$= \vec{0} + \omega\hat{k} \times R\hat{j}$$

$$= -\omega R\hat{i} = \frac{V_A R}{R-r}\hat{i}$$