Chapter 1

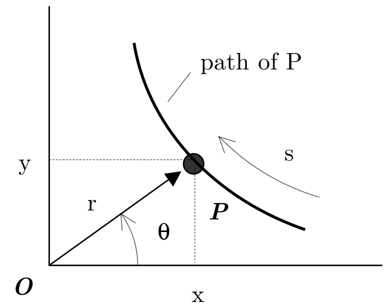
Particle Kinematics

A. Planar Kinematics: Cartesian, Path and Polar Coordinates

Chapter 1

Background

Point P moves on a curvilinear path in a plane. We will consider this planar motion of P using three different descriptions (see figure to the right):



- *Cartesian description* – here the path of P is known in terms of the Cartesian components x and y. Typically this path is given by an equation such as $y = y(x)$ that relates the x and y components.

- *Path description* – here the position is known in terms of a distance s measured along the path of the particle.

- *Polar description* – here the position is known in terms of a radial distance r (as measured from point O) and the angle θ for the line OP. The path of P is often expressed in terms of an equation such as $r = r(\theta)$ that relates r and θ.

Objectives

The goal of this lecture is to write the velocity and acceleration for the planar motion of a point in terms of three alternate descriptions: Cartesian, path and polar. The results of these three kinematic descriptions will be compared and contrasted exposing the attributes of each.

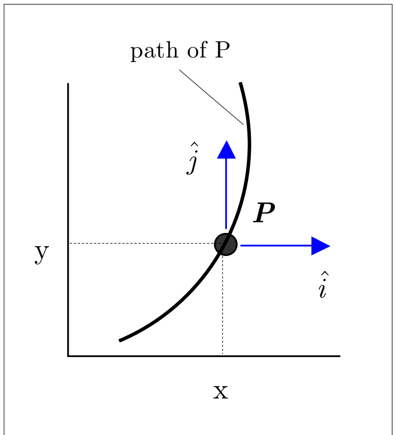
Lecture Material

As always, the velocity and acceleration of point P are given by the first and second time derivatives, respectively, of the position vector $\vec{r}$ for P:

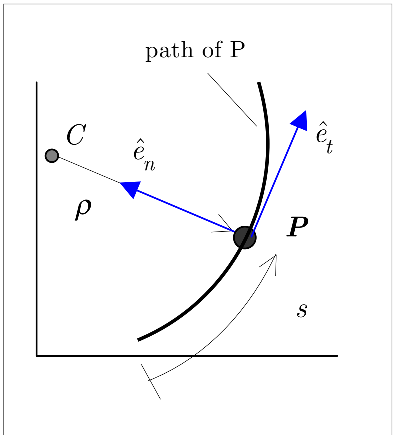
$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2}$$

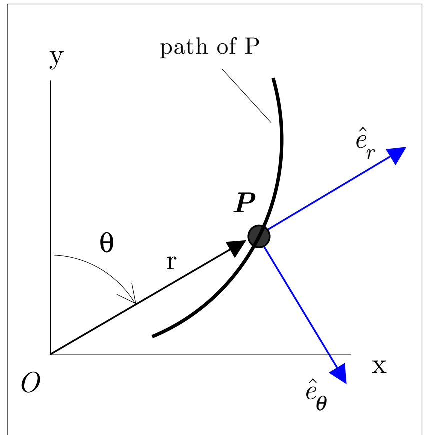
The kinematic equations for the Cartesian, path and polar descriptions are derived in the following notes. The following figures showing the kinematic variables and unit vectors will be used in these derivations.



Cartesian description



Path description



Polar description

Cartesian Kinematics

Here we write the position vector for P in terms of its x and y components and corresponding unit vectors:

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

Let's assume that $\hat{\imath}$ and $\hat{\jmath}$ represent constant directions (that is, $d\hat{\imath}/dt = \vec{0}$ and $d\hat{\jmath}/dt = \vec{0}$). Through differentiation with respect to time:

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

Discussion – Cartesian Description

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

From these equations, we see that the determination of the velocity and acceleration of a point in Cartesian components depends on our ability to differentiate the x and y components of its position with respect to time. Let's focus our attention to the difference between the EXPLICIT and IMPLICIT time dependence of these Cartesian components:

- If x and y are explicit functions of time, $x(t)$ and $y(t)$, then the Cartesian components for velocity and acceleration vectors are found directly by time differentiation of these functions.
- If the path of the point is given by the function $y = f(x)$, for example, and the kinematics are known for $x(t)$ (y is an implicit function of time; x is an explicit function of time), then we have to use the chain rule of differentiation. In this particular case, we have:

$$\dot{y} = \frac{dy}{dt} = \frac{d}{dt}f(x) = \frac{df}{dx}\frac{dx}{dt} = \dot{x}\frac{df}{dx}$$

Consider the following example:

MOTIVATING EXAMPLE Suppose that $y = \sin x$ and $\dot{x} = 3$ m/s = *constant*, and we want to know the velocity and acceleration when $x = \pi/2$. The Cartesian components of velocity and acceleration are given by:

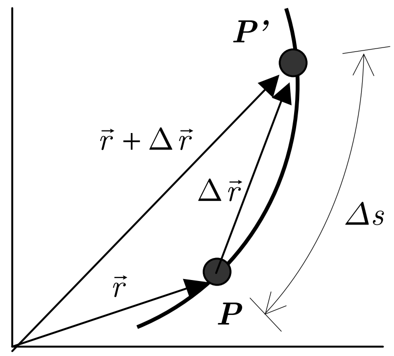
$$\begin{aligned}
\dot{y} &= \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \\
&= \dot{x}\frac{d}{dx}(\sin x) = \dot{x}\cos x \quad \Rightarrow \quad \dot{y} = (3)\cos\frac{\pi}{2} = 0 \text{ m/s} \\
\ddot{y} &= \frac{d}{dt}(\dot{x}\cos x) = \ddot{x}\cos x + \dot{x}(-\dot{x}\sin x) \\
&= \ddot{x}\cos x - \dot{x}^2\sin x \quad \Rightarrow \quad \ddot{y} = (0)\cos\frac{\pi}{2} - (3)^2\sin\frac{\pi}{2} = -9 \text{ m/s}^2
\end{aligned}$$

Therefore,

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath} = (3)\hat{\imath} + (0)\hat{\jmath} = 3\hat{\imath} \text{ m/s}$$
$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = (0)\hat{\imath} + (-9)\hat{\jmath} = -9\hat{\jmath} \text{ m/s}^2$$

Path Kinematics

Recall that for the path description, the position of particle P is given by a distance s measured along the path of P. The velocity of point P is given by the first time derivative of the position vector for P:



$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\frac{d\vec{r}}{ds}$$

where the *chain rule of differentiation* has been used to introduce the path distance s into the kinematics and where $v = ds/dt$ is the "speed" of the particle.

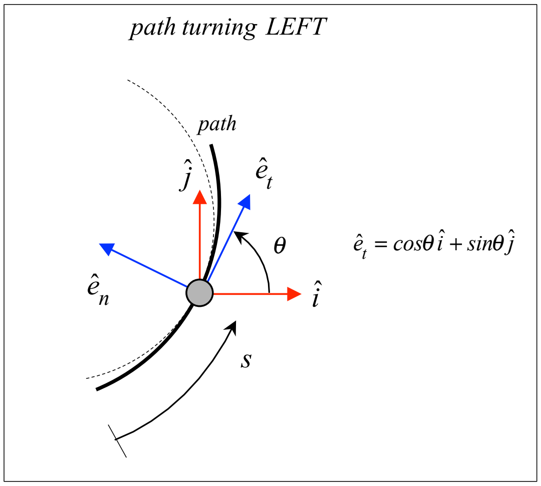
By definition, the derivative $\dfrac{d\vec{r}}{ds}$ is:

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\vec{r}}{\Delta s}$$

where $\Delta\vec{r}$ is the change in the position vector $\vec{r}$ as the particle moves a distance s along its path (as shown in the figure where P has moved to P').

Two observations on the above derivative, $\dfrac{d\vec{r}}{ds}$:

- As $\Delta s \to 0$, the chord length $|\Delta\vec{r}|$ tends to the length of the arc length Δs (see above figure): $|\Delta\vec{r}| \to \Delta s$. Therefore, $d\vec{r}/ds$ is a "unit vector" (magnitude of "1").
- As $\Delta s \to 0$, the vector $\Delta\vec{r}$ becomes tangent to the path of P (see above figure).



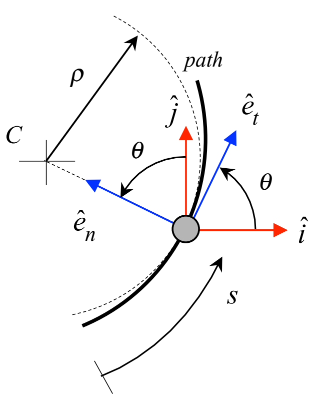
From this we conclude that $d\vec{r}/ds = \hat{e}_t$ = unit vector that is tangent to the path of P. Therefore,

$$\vec{v} = v\hat{e}_t$$

Differentiation of the above with respect to time gives:

$$
\begin{aligned}
\vec{a} &= \frac{d\vec{v}}{dt} \\
&= \frac{d}{dt}\left(v\hat{e}_t\right) \\
&= \frac{dv}{dt}\hat{e}_t + v\frac{d\hat{e}_t}{dt} \qquad \text{(product rule of differentiation)} \\
&= \dot{v}\hat{e}_t + v\frac{d\hat{e}_t}{ds}\frac{ds}{dt} \qquad \text{(chain rule of differentiation)} \\
&= \dot{v}\hat{e}_t + v^2\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} \qquad \text{(chain rule of differentiation and } v = \frac{ds}{dt}\text{)}
\end{aligned}
$$

path turning LEFT



$$\hat{e}_n = -sin\theta\,\hat{i} + cos\theta\,\hat{j}$$

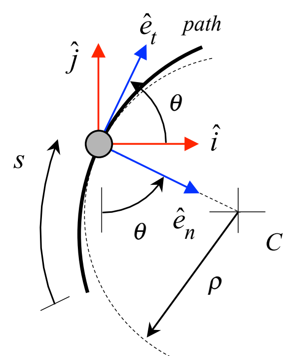
$$\hat{e}_t = cos\theta\,\hat{i} + sin\theta\,\hat{j} \quad \Rightarrow$$

$$\frac{d\hat{e}_t}{d\theta} = -sin\theta\,\hat{i} + cos\theta\,\hat{j} = \hat{e}_n$$

$$ds = +\rho d\theta \quad \Rightarrow \quad \frac{d\theta}{ds} = \frac{1}{\rho}$$

$$\therefore \quad \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

path turning RIGHT



$$\hat{e}_n = sin\theta\,\hat{i} - cos\theta\,\hat{j}$$

$$\hat{e}_t = cos\theta\,\hat{i} + sin\theta\,\hat{j} \quad \Rightarrow$$

$$\frac{d\hat{e}_t}{d\theta} = -sin\theta\,\hat{i} + cos\theta\,\hat{j} = -\hat{e}_n$$

$$ds = -\rho d\theta \quad \Rightarrow \quad \frac{d\theta}{ds} = -\frac{1}{\rho}$$

$$\therefore \quad \frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} = \frac{\hat{e}_n}{\rho}$$

$C =$ "*center of curvature*" (*center of circle tangent to path having the same curvature as the path*)

$\rho =$ "*radius of curvature*" *of path*

Consider the figures provided above showing the directions of the unit vectors $\hat{e}_t$ and $\hat{e}_t$ for left-turning and right-turning paths. From these figures, we see that for both cases:

$$\frac{d\hat{e}_t}{d\theta}\frac{d\theta}{ds} = \frac{1}{\rho}\hat{e}_n$$

where ρ is the radius of curvature of the path. Using this relationship in the above gives:

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

which describes the acceleration of a particle in terms of its path components.

Discussion – Path Description

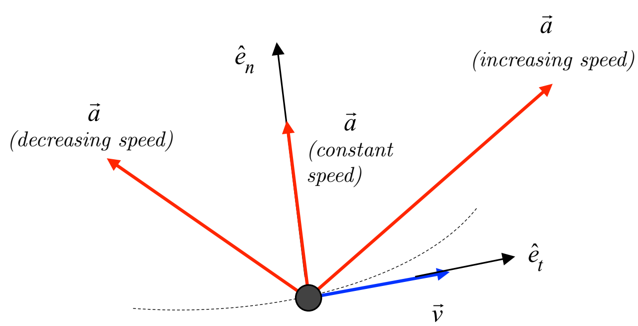
We have seen that the velocity and acceleration of a particle can be written in terms of its path components by the following equations:

$$\vec{v} = v\hat{e}_t$$

$$\vec{a} = \dot{v}\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n$$

- The velocity of a point is ALWAYS tangent to the path of the point. The magnitude of the velocity vector is the known as the scalar "speed" v of the point.

- The acceleration of the point has two components:
  - The component $\left(v^2/\rho\right)\hat{e}_n$ is *normal* to the path. This is commonly referred to as the "centripetal" component of acceleration. This component is ALWAYS directed inward to the path (*positive n-component*) since $v^2/\rho > 0$.
  - The component $\dot{v}\hat{e}_t$ is tangent to the path. The magnitude of this component is the "rate of change of speed" $\dot{v}$ for the point.
    * When $\dot{v} > 0$ (increasing speed), the acceleration vector has a *positive t-component* (i.e., forward of $\hat{e}_n$).
    * When $\dot{v} = 0$ (constant speed), the acceleration vector has a *zero t-component* (i.e., $\vec{a}$ is aligned with $\hat{e}_n$). Note that constant speed does NOT imply zero acceleration!
    * When $\dot{v} < 0$ (decreasing speed), the acceleration vector has a *negative t-component* (i.e., backward of $\hat{e}_n$). See figure below.



- ρ is the radius of curvature for the path of the particle. If the path is known to be circular, ρ is the radius of the circle. For a general path known in terms of its Cartesian coordinates $y = y(x)$, the radius of curvature can be calculated from:

$$\rho = \frac{\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\left|\dfrac{d^2y}{dx^2}\right|}$$

- The magnitude of the acceleration is given by the square root of the sum of the squares of its path components:

$$|\vec{a}| = \sqrt{\dot{v}^2 + (v^2/\rho)^2}$$

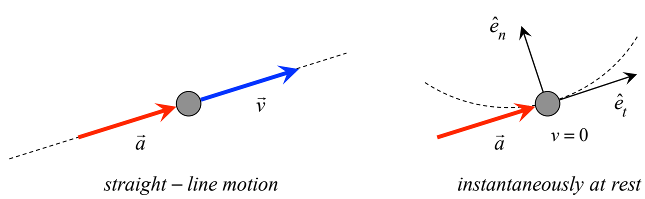
  Do not confuse the terms "rate of change of speed" and "magnitude of acceleration":

  - Rate of change of speed $\dot{v}$ (as the name indicates) is the rate at which the speed changes in time; it is simply the tangential component of acceleration.

  - The magnitude of acceleration $|\vec{a}|$ accounts for both the tangential and normal components of acceleration, as shown in the above equation.

CHALLENGE QUESTION: The acceleration vector is the time derivative of the velocity vector: $\vec{a} = d\vec{v}/dt$. In contrast, the scalar rate of change of speed is the time derivative of the scalar speed: $\dot{v} = dv/dt$. As discussed above, the magnitude of acceleration is generally not the same as the magnitude of the rate of change of speed: $|\vec{a}| \neq |\dot{v}|$. Are there situations in which they are the same?

ANSWER: Since $|\vec{a}| = \sqrt{\dot{v}^2 + (v^2/\rho)^2}$, $|\vec{a}| = |\dot{v}|$ ONLY IF $v^2/\rho = 0$. This occurs when either: (i) $\rho = \infty$ (straight-line, or rectilinear, motion), or (ii) $v = 0$ (particle instantaneously at rest). These two situations are shown in the following figure.



Polar Kinematics

For the polar description, the following set of unit vectors will be used:

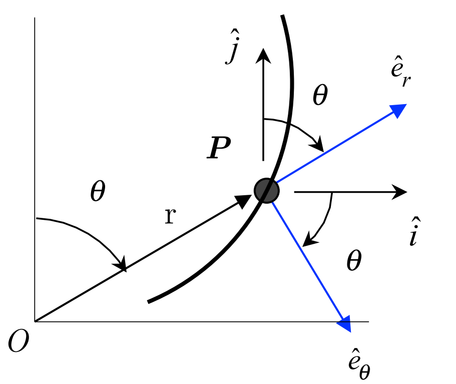
- $\hat{e}_r$: pointing from O to point P
- $\hat{e}_\theta$: perpendicular to $\hat{e}_r$ and pointing in the "positive θ direction" (see the figure below)

Here we write the position vector of P as:

$$\vec{r} = r\hat{e}_r$$

The velocity of point P is given by the first time derivative of the position vector for P:

$$\begin{aligned}\vec{v} &= \frac{d}{dt}\left(r\hat{e}_r\right) = \frac{dr}{dt}\hat{e}_r + r\frac{d\hat{e}_r}{dt} && \text{(product rule of differentiation)}\\ &= \dot{r}\hat{e}_r + r\frac{d\hat{e}_r}{d\theta}\frac{d\theta}{dt} && \text{(chain rule of differentiation)}\end{aligned}$$



$$\begin{aligned}\hat{e}_r &= \sin\theta\,\hat{i} + \cos\theta\,\hat{j}\\ \hat{e}_\theta &= \cos\theta\,\hat{i} - \sin\theta\,\hat{j}\end{aligned} \quad \Rightarrow$$

$$\frac{d\hat{e}_r}{d\theta} = \cos\theta\,\hat{i} - \sin\theta\,\hat{j} = \hat{e}_\theta$$

$$\frac{d\hat{e}_\theta}{d\theta} = -\sin\theta\,\hat{i} - \cos\theta\,\hat{j} = -\hat{e}_r$$

Using the equations in the figure above, we can now write the velocity vector for P as:

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

Using the product rule of differentiation on the velocity vector above, we obtain:

$$\vec{a} = \frac{d}{dt}\left(\dot{r}\hat{e}_r\right) + \frac{d}{dt}\left(r\dot{\theta}\hat{e}_\theta\right) = \ddot{r}\hat{e}_r + \dot{r}\frac{d}{dt}\left(\hat{e}_r\right) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\frac{d}{dt}\left(\hat{e}_\theta\right)$$

From above, we know that $\frac{d}{dt}\left(\hat{e}_r\right) = \dot{\theta}\hat{e}_\theta$, and through the use of the chain rule and the figure above, it can be shown that:

$$\frac{d}{dt}\left(\hat{e}_\theta\right) = \frac{d\hat{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{e}_r$$

Therefore, we have:

$$\begin{aligned}\vec{a} &= \ddot{r}\hat{e}_r + \dot{r}\left(\dot{\theta}\hat{e}_\theta\right) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\left(-\dot{\theta}\hat{e}_r\right) \\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta\end{aligned}$$

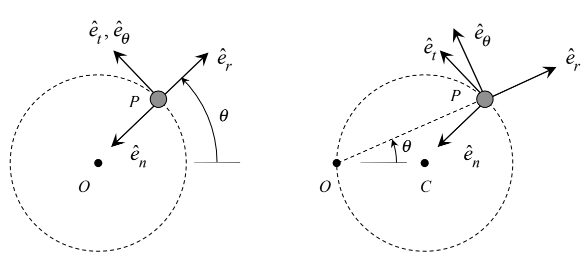
Discussion – Polar Description

$$\vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$
$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta$$

- *The values for the components of these vectors depend on your choice of the point O.* Therefore, you need to carefully define your choice of point O at the beginning of the problem and stick with it throughout the problem.
- When the path of P is given as $r = r(\theta)$ you will need to use the *chain rule of differentiation* to find the time derivatives $\dot{r} = dr/dt$ and $\ddot{r} = d^2r/dt^2$ in terms of the time derivatives $\dot{\theta}$ and $\ddot{\theta}$.
- These vector expressions use the components of velocity and acceleration projected on the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$. These projections are usually more difficult to determine than, say, the Cartesian projections and have less physical significance than the path component projections. However, in many applications involving observers of motion, the polar expressions are very useful.

CHALLENGE QUESTION: The path unit vectors ($\hat{e}_t$ and $\hat{e}_n$) share characteristics with the polar unit vectors ($\hat{e}_r$ and $\hat{e}_\theta$) in that they move along with the particle and they change orientation as the particle moves along its path. Can the two sets of unit vectors ever be aligned with each other?

ANSWER: As we know, the path unit vectors $\hat{e}_t$ and $\hat{e}_n$ are defined by the path. On the other hand, the polar unit vectors $\hat{e}_r$ and $\hat{e}_\theta$ depend on your choice of the observer O and the position of the particle relative to O. One special case when they are somewhat aligned is when the particle travels on a circular path with the observer O at the center of the circle, as shown in the figure below left. Here, $\hat{e}_t$ and $\hat{e}_\theta$ are aligned for all motion; $\hat{e}_r$ points outward from O and $\hat{e}_n$ points inward toward the center of the path O. To emphasize how the orientation of $\hat{e}_r$ and $\hat{e}_\theta$ depends on the choice of O, consider moving O to another location, as shown below right. Here the two sets of unit vectors are not aligned.

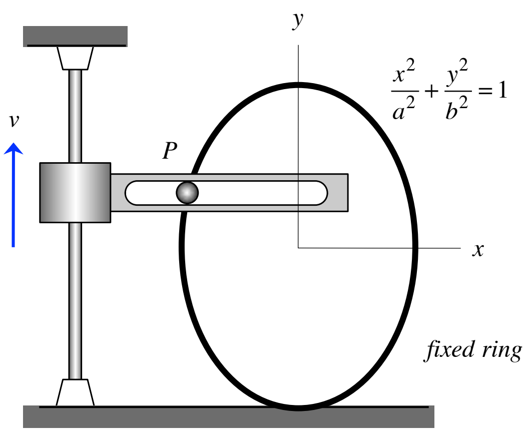


Example 1.A.1

Given: Pin P is constrained to move along a elliptical ring whose shape is given by $x^2/a^2+y^2/b^2 = 1$ (where x and y are given in mm). The pin is also constrained to move within a horizontal slot that is moving upward at a constant speed of v.

Find: Determine:
(a) The velocity of pin P at the position corresponding to $y = 6$ mm; and
(b) The acceleration of pin P at the position corresponding to $y = 6$ mm.

Use the following parameters in your analysis: $a = 5$ mm, $b = 10$ mm, $v = 30$ mm/s.



Example 1.A.2

Given: A particle P moves on a path whose Cartesian components are given by the following functions of time (where both components are given in inches and time t is given in seconds):

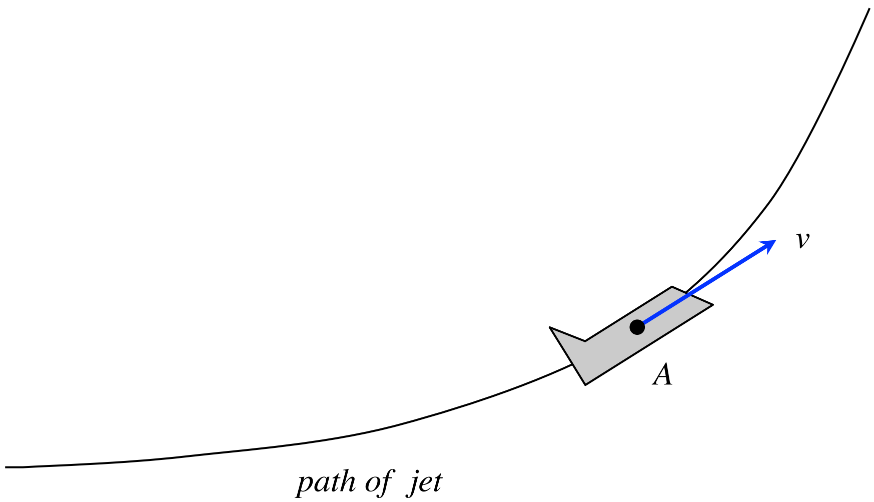
$$x(t) = t^3 + 10$$
$$y(t) = 2\cos 4t$$

Find: Determine at the time $t = 2$ s:
(a) The velocity vector of P;
(b) The acceleration of P; and
(c) The angle between the velocity and acceleration vectors of P.

Example 1.A.3

Given: A jet is flying on the path shown below with a speed of v. At position A on the loop, the speed of the jet is $v = 600$ km/hr, the magnitude of the acceleration is 2.5g and the tangential component of acceleration is $a_t = 5$ m/s^2.

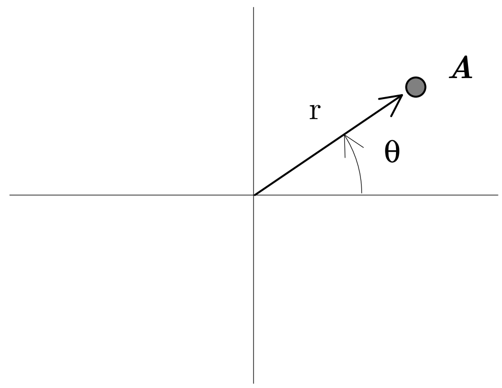
Find: The radius of curvature of the path of the jet at A.



Example 1.A.4

Given: Particle A travels on a path such that the radial position of A is given by $r = 5\theta$, where r is given in meters and θ in radians. It is also known that $\dot{\theta} = 2$ rad/s = *constant*.

Find: Determine:
(a) The velocity vector for A when $\theta = \pi$; and
(b) The acceleration vector for A when $\theta = \pi$.



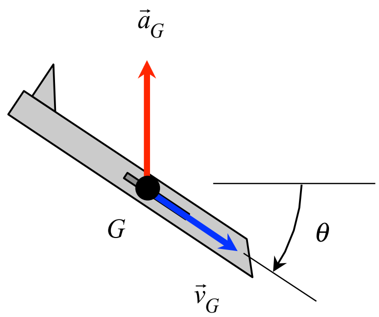
Example 1.A.5

Given: At one instant in time, an aircraft is traveling along a path in a direction defined by θ below the horizontal with the center of mass G of the aircraft having a speed of $|\vec{v}_G|$. G is also known to have an acceleration that is pointing vertically upward with a magnitude of $|\vec{a}_G|$.

Find: For this given instant in time:

(a) show the path unit vectors $\hat{e}_t$ and $\hat{e}_n$, along with $\vec{v}_G$ and $\vec{a}_G$, in a sketch.

(b) determine the rate of change of speed of G and the radius of curvature of G.



Use the following parameters in your work: $\theta = 36.87°$, $|\vec{v}_G| = 900$ km/hr and $|\vec{a}_G| = 30$ m/s^2.

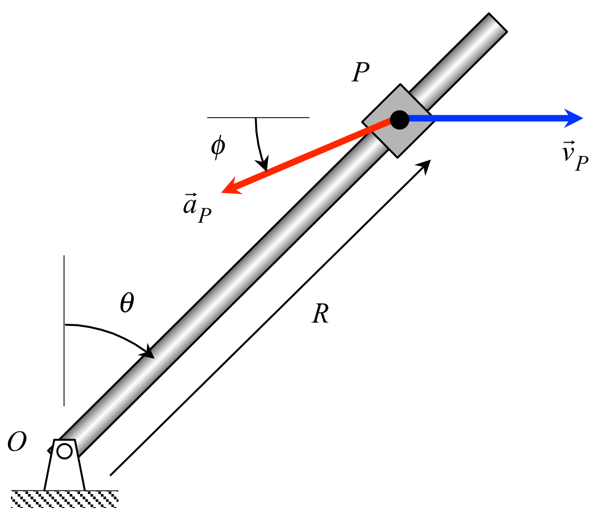
Example 1.A.6

Given: Particle P is able to slide along an arm that is rotating about end O. At the instant shown, the arm is at an angle of θ measured clockwise from the vertical, the velocity of P is known to be horizontal, and the acceleration of P is in a direction defined by the angle ϕ from the horizontal, all as shown in the figure.

Find: For position of $\theta = 30°$:

(a) show the position of P and the polar unit vectors $\hat{e}_R$ and $\hat{e}_\theta$, along with $\vec{v}_P$ and $\vec{a}_P$, in a sketch.

(b) determine numerical values for $\dot{R}$, $\ddot{R}$, $\dot{\theta}$ and $\ddot{\theta}$.



Use the following parameters in your work: $R = 2\text{m}$, $|\vec{v}_P| = 8$ m/s and $|\vec{a}_P| = 20$ m/s^2.