Lecture 31: Stress transformation – Principal stresses and maximum shear stress

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Fall 2019

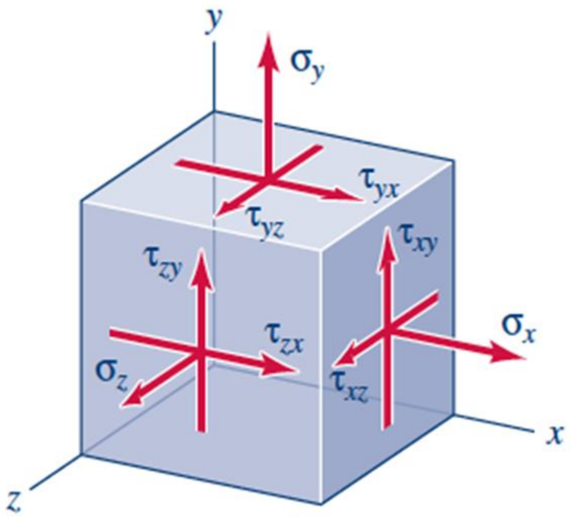
Lecture Book: Chapter 13

PURDUE ENGINEERING

Motivation

- There is only *one* state of stress at a point
- BUT we can express that state of stress in different coordinate systems
  - A material could fail on a surface that does *not* correspond to our *x*, *y*, or *z* axes

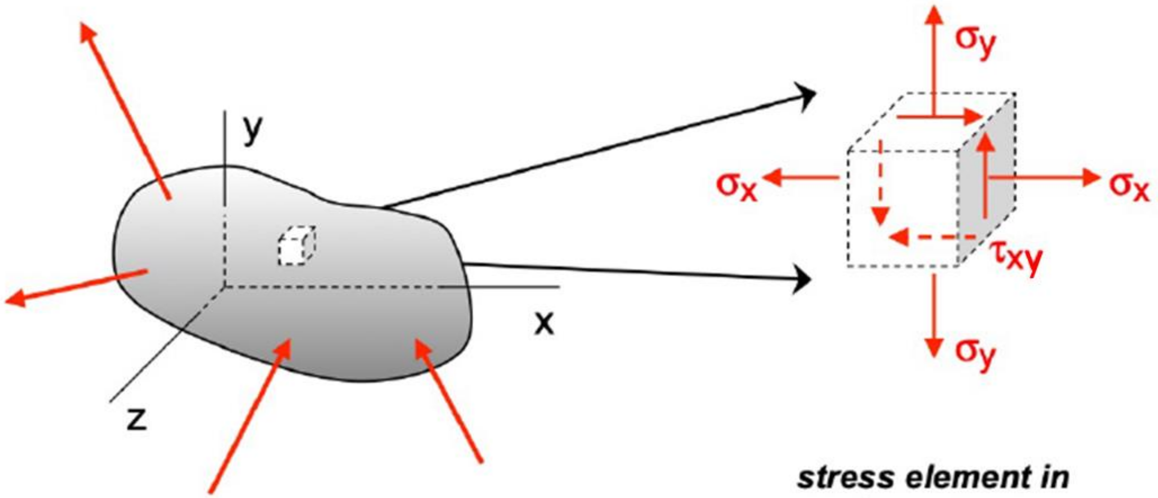
<u>Recall a general 3-D state of stress (Chapter 5)</u>



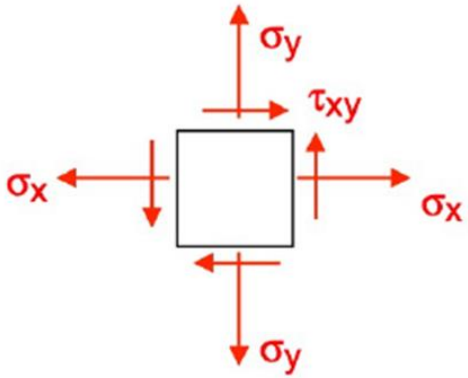
States of stress: Plane stress

Many engineering components are in a state of "plane stress"

For plane stress in the *x-y* plane: $\sigma_x \neq 0,\ \sigma_y \neq 0,\ \tau_{xy} \neq 0 \qquad \sigma_z = 0,\ \tau_{xz} = 0,\ \tau_{yz} = 0$



stress element in plane stress



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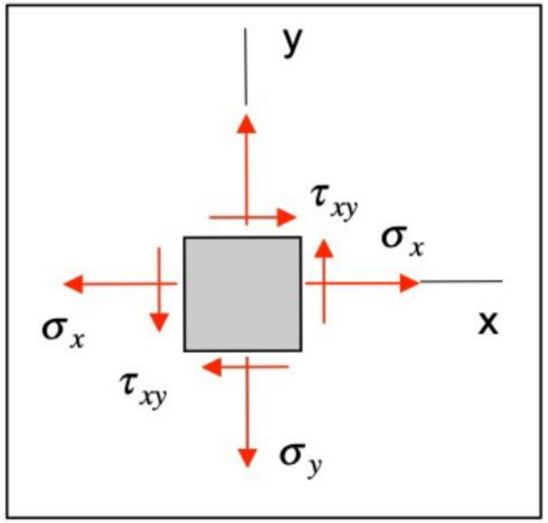
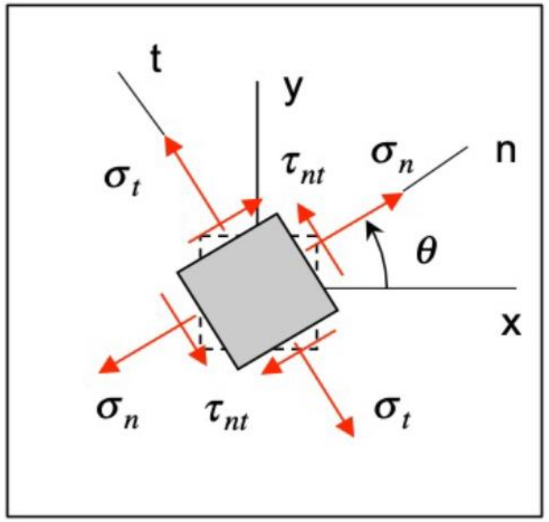
BEFORE rotation



AFTER rotation

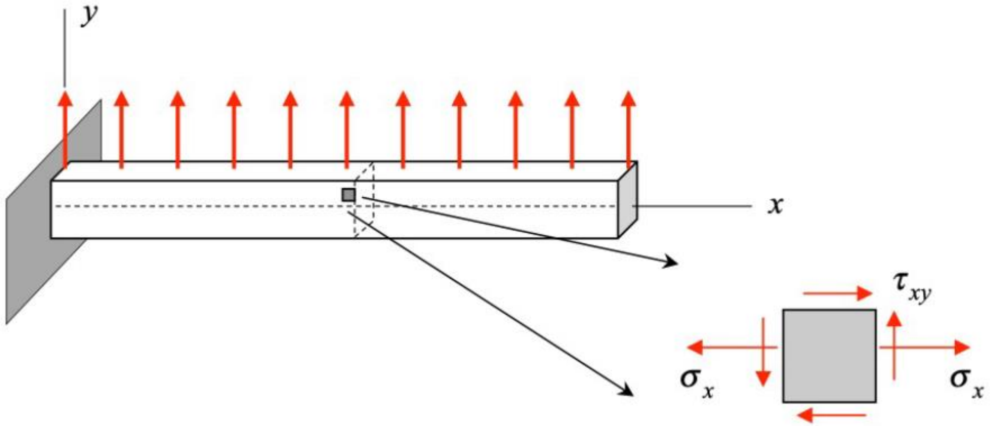


What if we want to express this state of stress with respect do different coordinates?

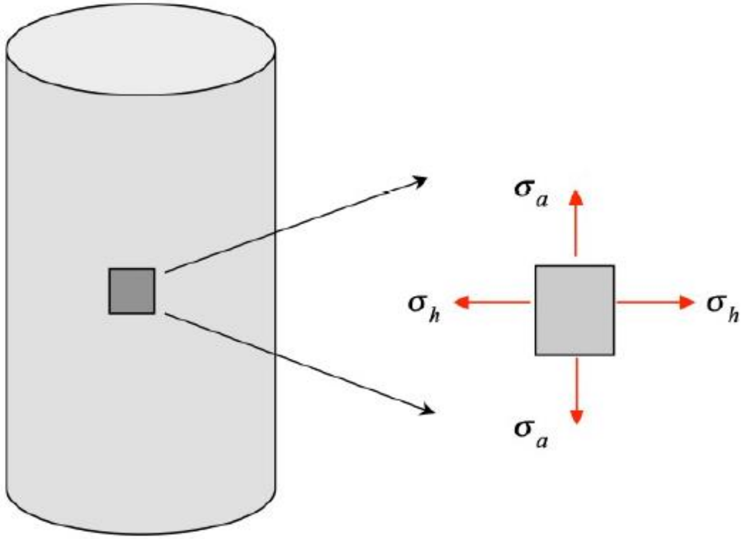
Plane stress examples

Each type of deformation we have looked at so far, along with combinations of each of them

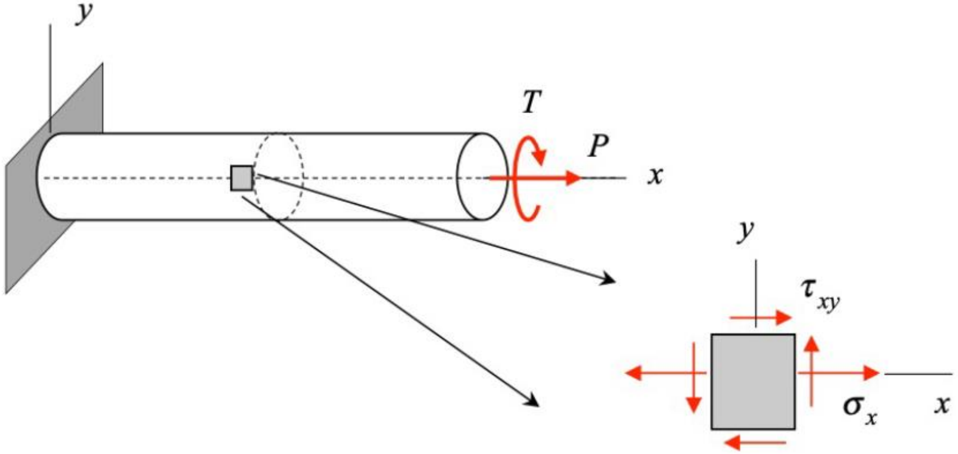
BENDING BEAM



THIN-WALLED PRESSURE VESSEL



TORSIONALLY AND AXIALLY LOADED SHAFT



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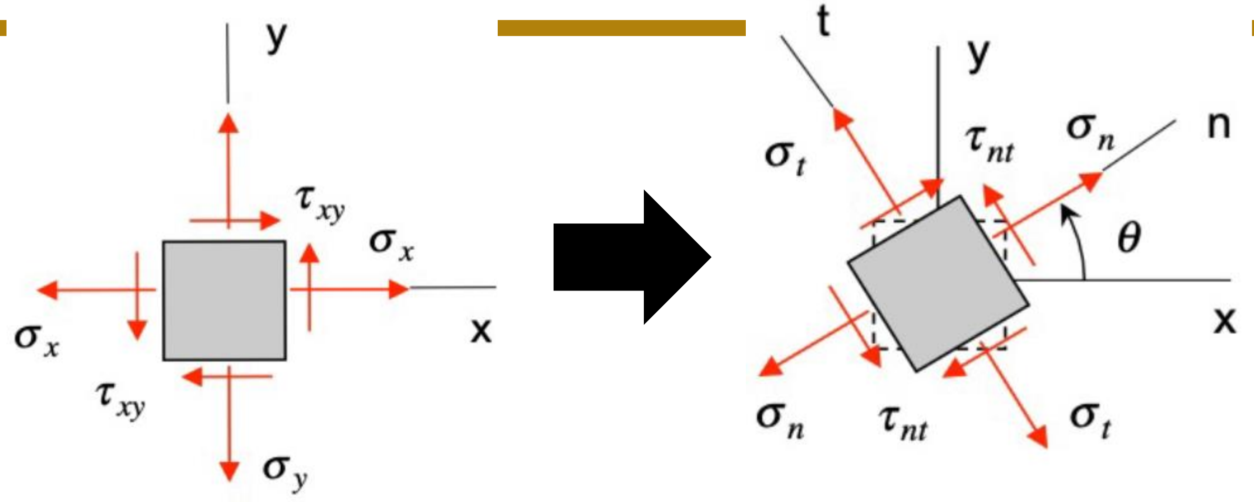
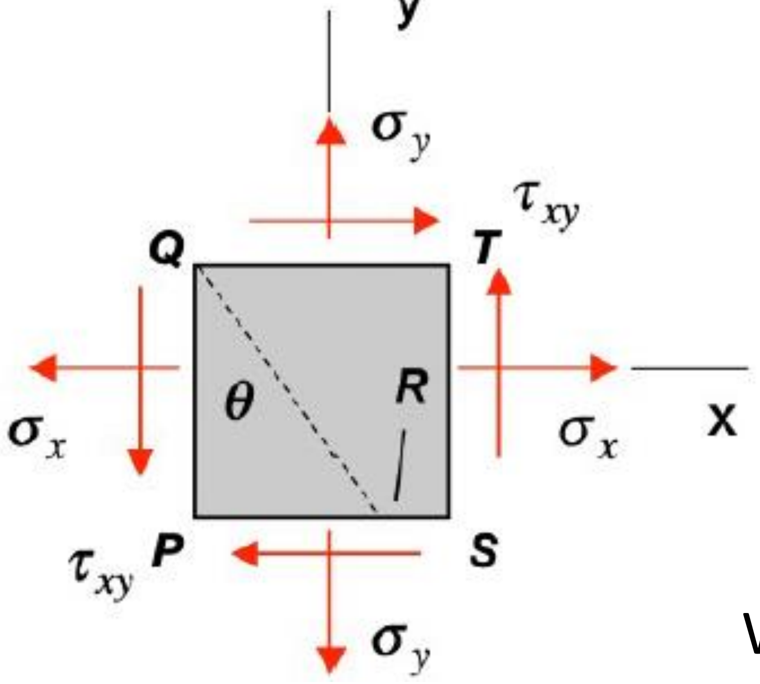
Stress transformation for plane stress

Goal: Find an expression for $\sigma_n, \sigma_t, \tau_{nt}$ from $\sigma_x, \sigma_y, \tau_{xy}$



To do this, enforce equilibrium on the following cut:



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We end up with:

$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\left(2\sin\theta\cos\theta\right)$$

$$\tau_{nt} = -\left(\sigma_x - \sigma_y\right)\cos\theta\sin\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

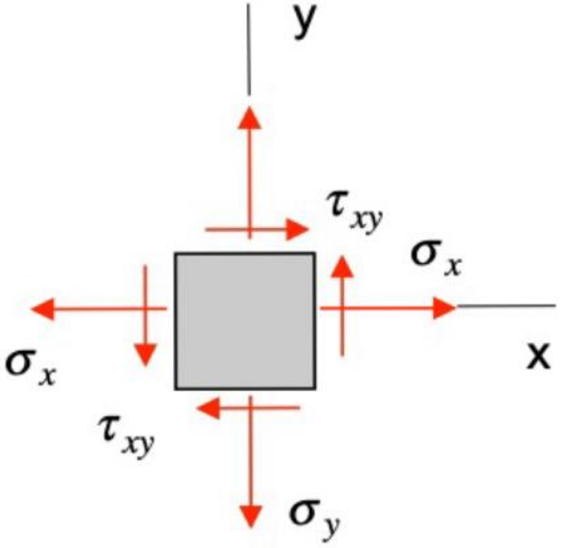
Stress transformation for plane stress

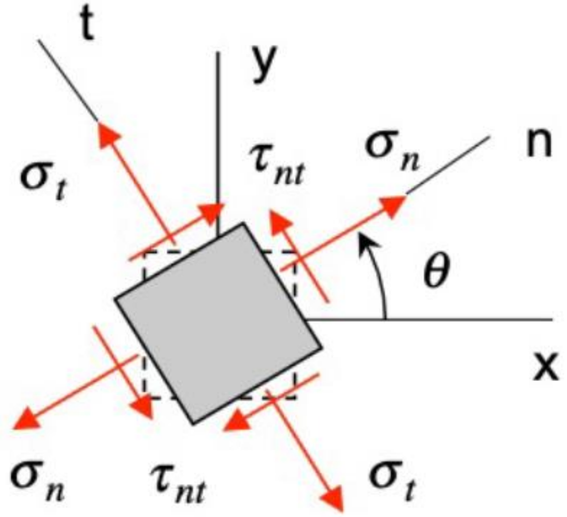
Stress transformation equations for plane stress

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_t = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$





θ measured CCW from *x* axis

$\sigma > 0$ for tension

$\tau_{ij} > 0$ for shear stress on the positive *i* face in the positive *j* direction

Remember: each set of stresses ($\sigma_x, \sigma_y, \tau_{xy}$ and $\sigma_n, \sigma_t, \tau_{nt}$) represent the *same* stress state, just with respect to different coordinate axes

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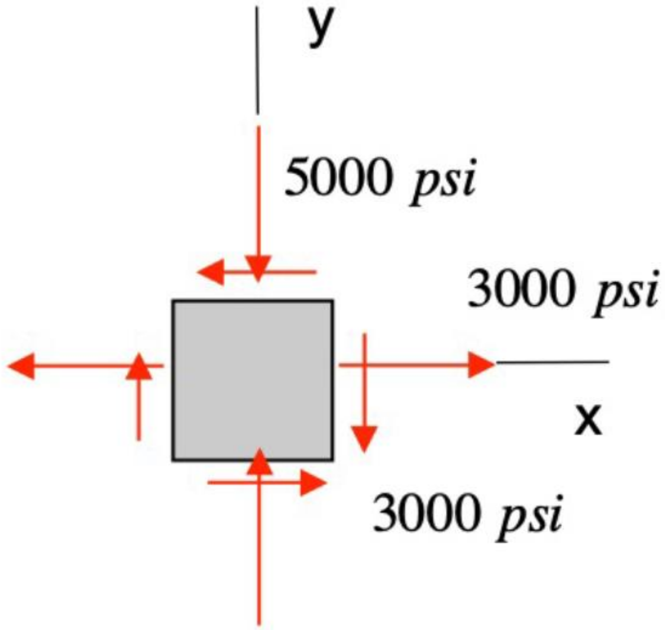
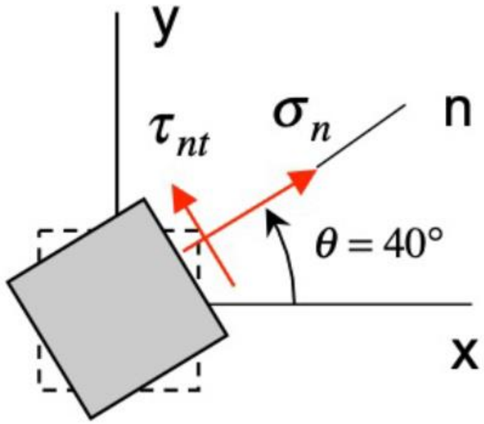
Example 13.1

Determine the stresses on a plane whose orientation is a 40^{o} CCW rotation from the x-axis



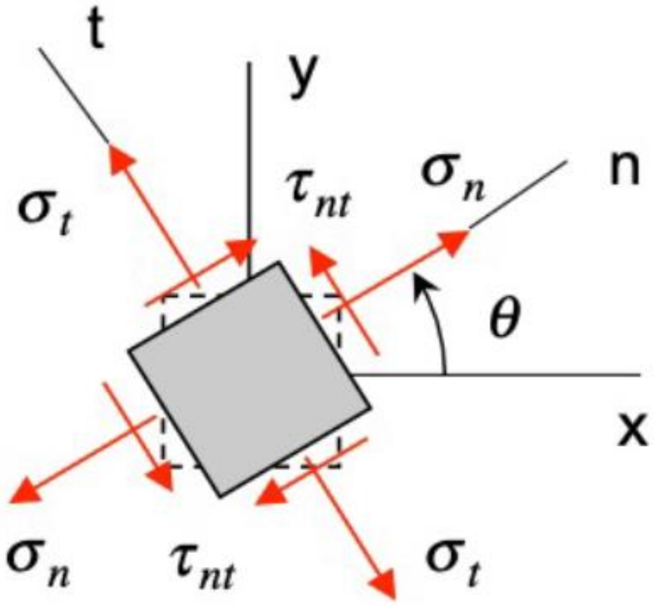


Special states of stress

We want to predict *failure* of a component

→ we are often interested in the *maximum normal stresses* and *maximum shear stress*

The maximum normal stresses are called <u>principal stresses</u>



From the stress transformation equation, how do we find the extrema of σ_n ?

$$\sigma_n = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

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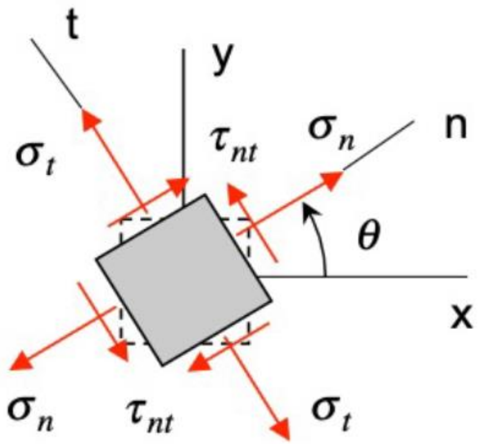
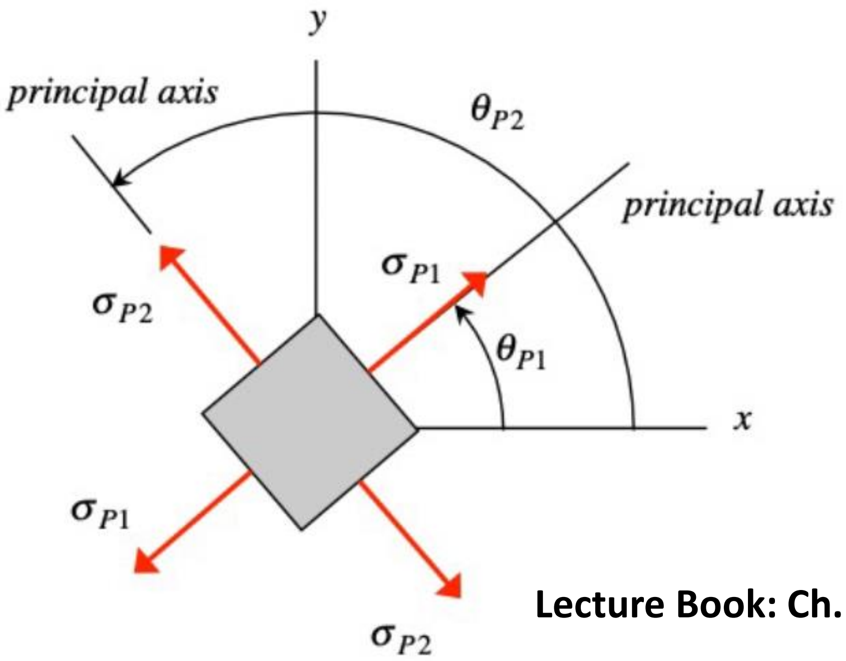
Principal stresses





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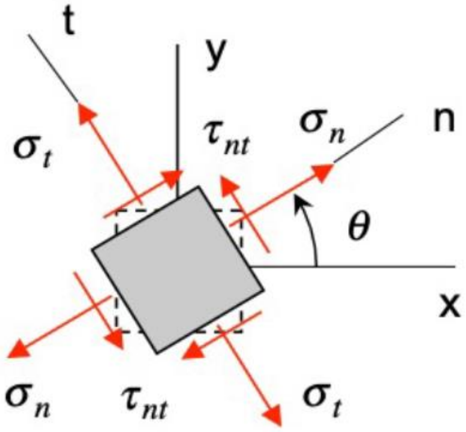
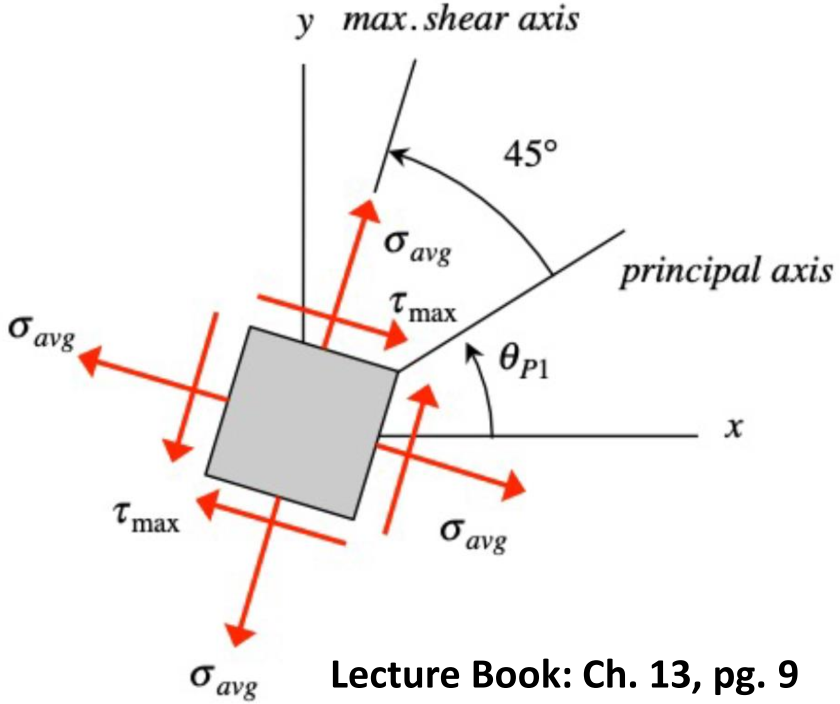
Maximum in-plane shear stress



From the stress transformation equation, how do we find the extrema of τ_{nt} ?

$$\tau_{nt} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$



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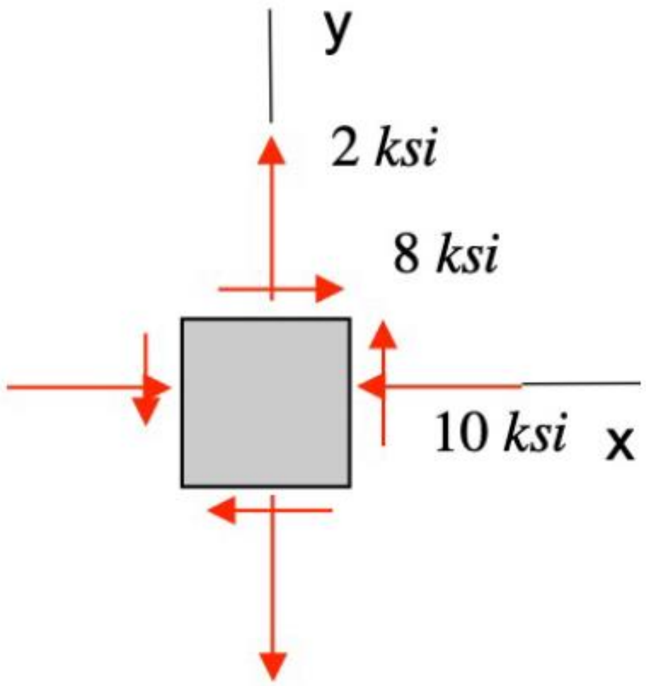
Example 13.2

For the stress state shown, determine:

-The in-plane principal stresses and their orientation

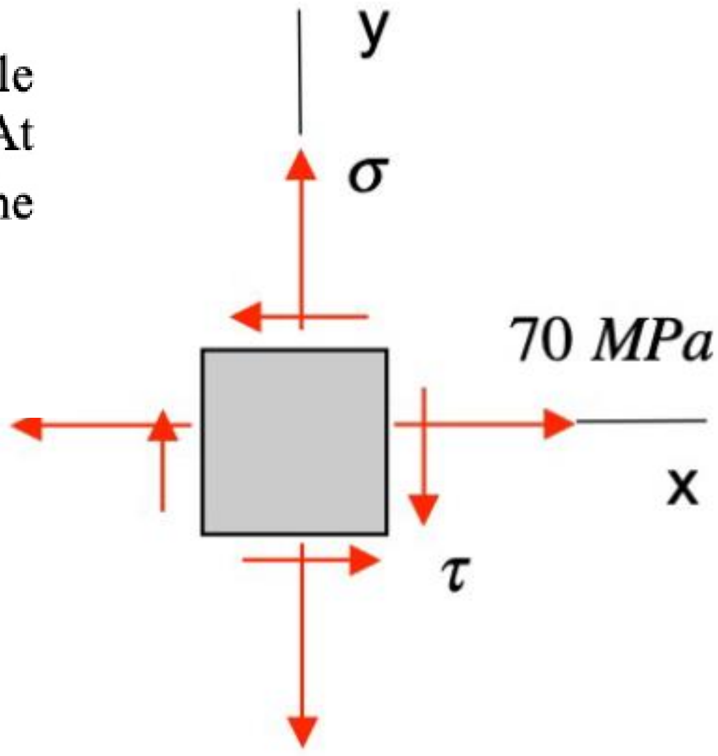
-The maximum in-plane shear stress and its orientation



Example 13.3

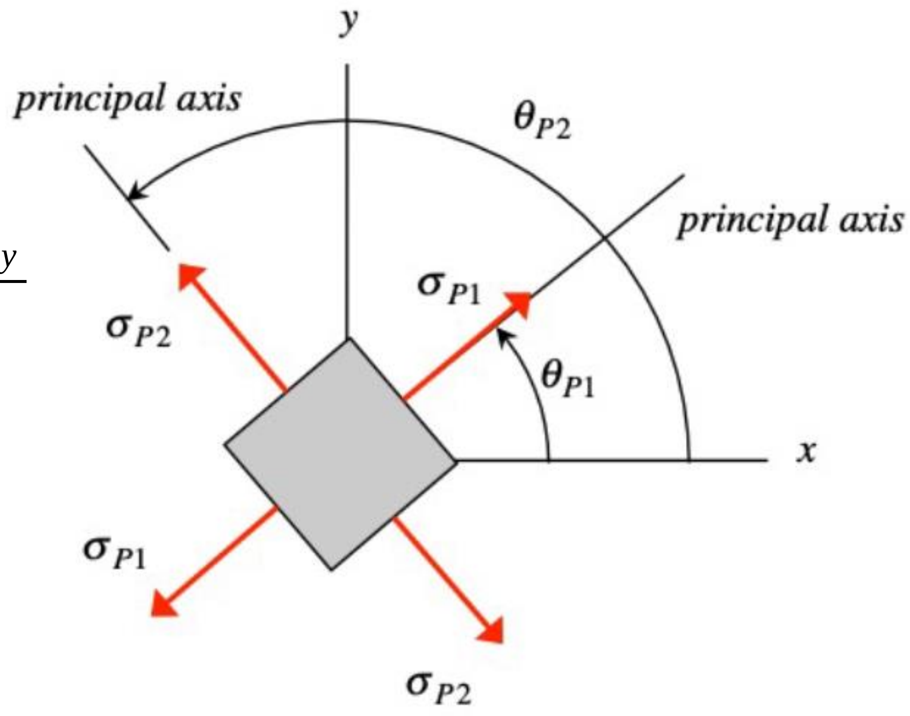
The state of plane stress at a point shown below can be described by a known tensile stress $\sigma_x = 70 MPa$, and unknown tensile stress σ and an unknown shear stress τ. At this point, the maximum in-plane shear stress is known to be 78 *MPa*, and one of the two in-plane principal stresses is 22 *MPa* (in tension).

Determine the values of σ and τ, as well as the other in-plane principal stress.



Summary for principal stresses

We can use the following equations to calculate the principal stresses and directions given $\sigma_x, \sigma_y, \tau_{xy}$



Maximum in-plane normal stress $\quad \sigma_{p1} = \sigma_{ave} + R \qquad \sin\left(2\theta_{p1}\right) = \dfrac{\tau_{xy}}{R} \qquad \cos\left(2\theta_{p1}\right) = \dfrac{\sigma_x - \sigma_y}{2R}$

Minimum in-plane normal stress $\quad \sigma_{p2} = \sigma_{ave} - R \qquad \theta_{p2} = \theta_{p1} + 90^{o}$

where $\quad \sigma_{ave} = \dfrac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\dfrac{\left(\sigma_x - \sigma_y\right)^2}{4} + \tau_{xy}{}^2}$

The shear stress is zero on the principal planes! $\tau_{nt}\left(\theta_{p1}\right) = \tau_{nt}\left(\theta_{p2}\right) = 0$

Summary for maximum in-plane shear stress

We can use the following equations to calculate the maximum and minimum in-plane shear stress given $\sigma_x, \sigma_y, \tau_{xy}$

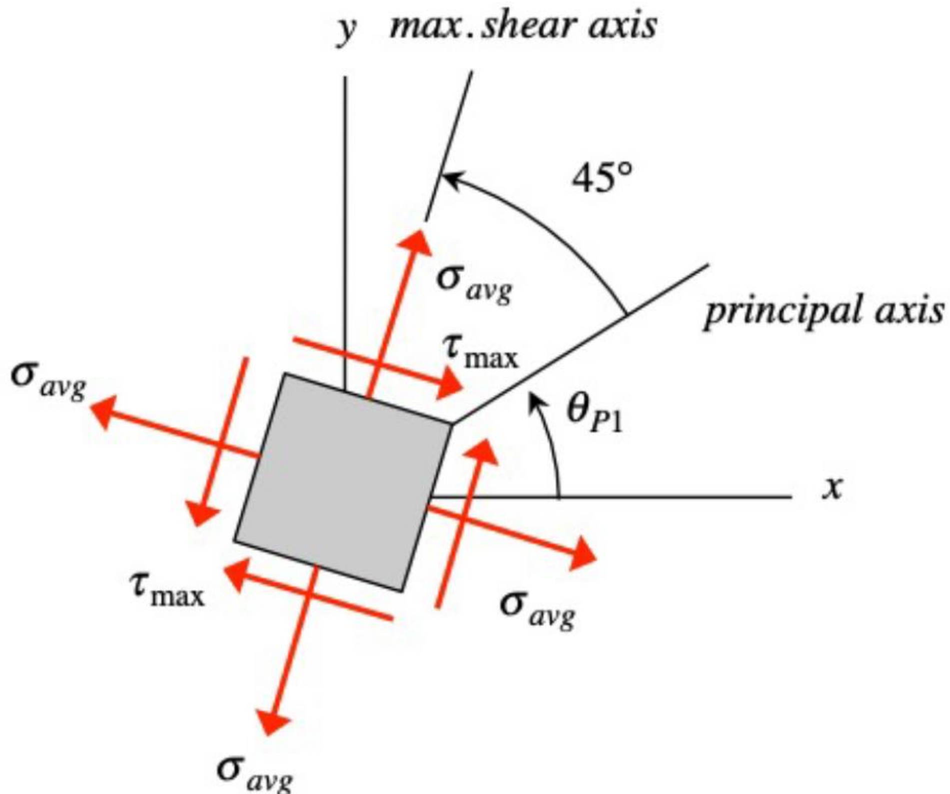
Maximum in-plane shear stress

$$\tau_{max} = \tau_{s1} = R \qquad \theta_{s1} = \theta_{p1} + 45^{o} \qquad \sin(2\theta_{s1}) = -\frac{\sigma_x - \sigma_y}{2R} \qquad \cos(2\theta_{s1}) = \frac{\tau_{xy}}{R}$$

Minimum in-plane shear stress

$$\tau_{min} = \tau_{s2} = -R = -\tau_{max} \qquad \theta_{s2} = \theta_{s1} + 90^{o}$$

$$\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}{}^2}$$

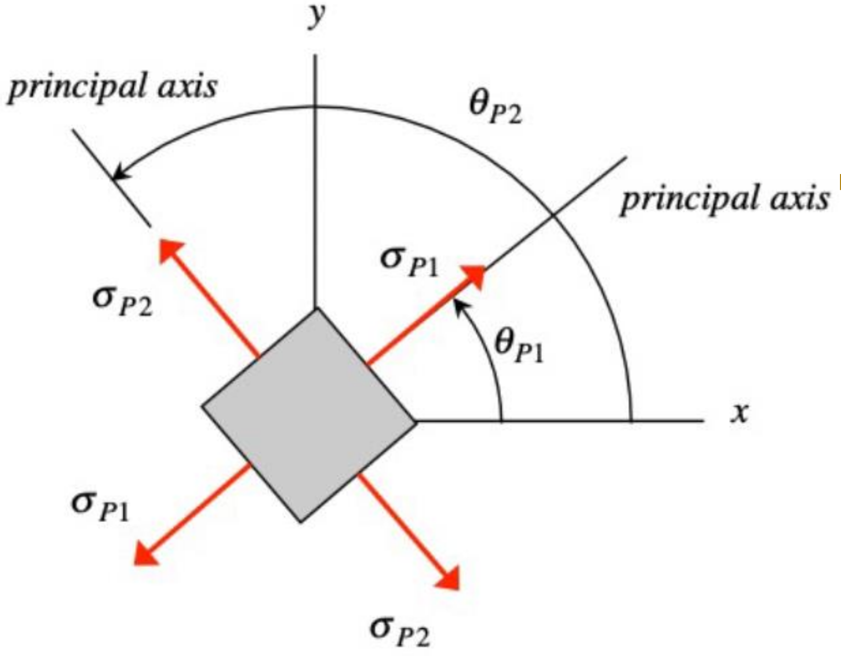


Both normal stresses are equal to the *average* normal stress in a maximum or minimum shear stress state

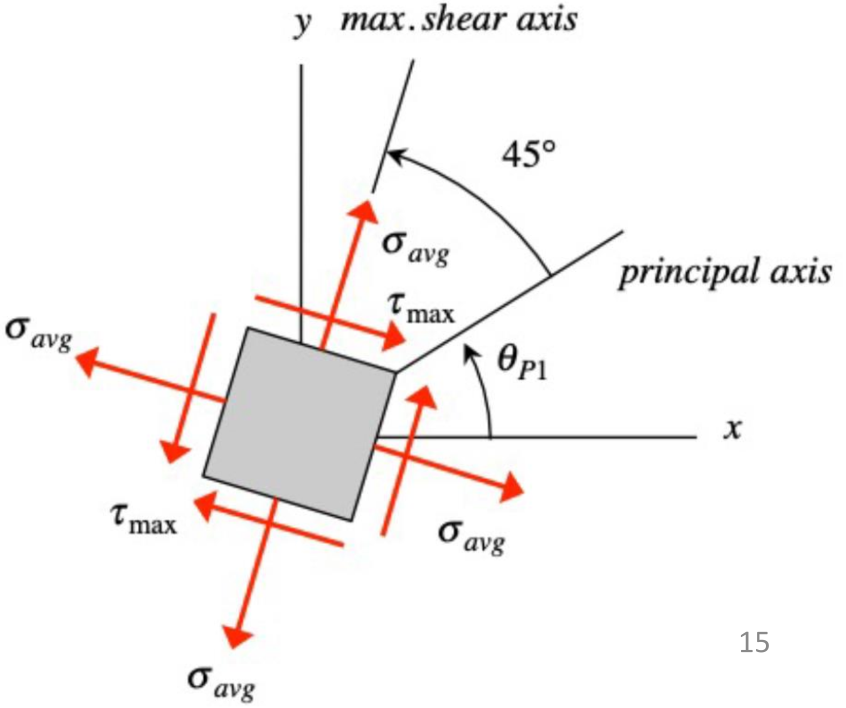
$$\sigma_n(\theta_{s1,s2}) = \sigma_t(\theta_{s1,s2}) = \sigma_{avg}$$

Summary

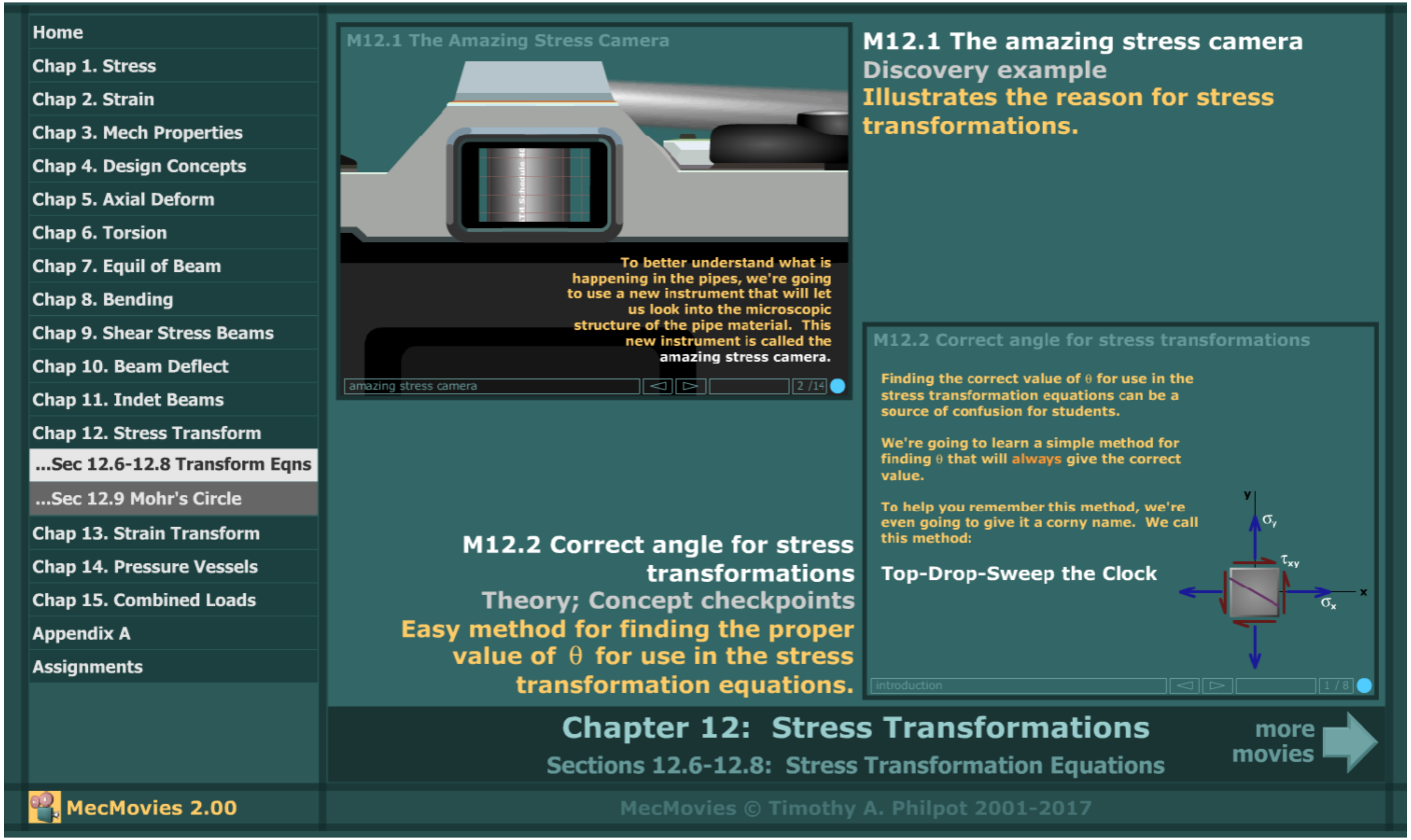
- The in-plane principal stresses are:
  - The max and min in-plane normal stresses
  - Found at orientations with no shear stress
  - Always on planes 90^o apart



- The max and min in-plane shear stresses are:
  - Equal in magnitude
  - Found at orientations where the normal stresses are both equal to $\sigma_{ave} = \left(\sigma_x + \sigma_y\right) / 2$
  - Always on planes 45^o away from principal stress planes



Visualizing stress transformations



"Mec movies" – see M12.1 for help visualizing stress transformations, principal stresses, and maximum shear stress

http://web.mst.edu/~mecmovie/