Lecture 40-41: Failure analysis (static failure theories)

Joshua Pribe

Fall 2019

Lecture Book: Ch. 15

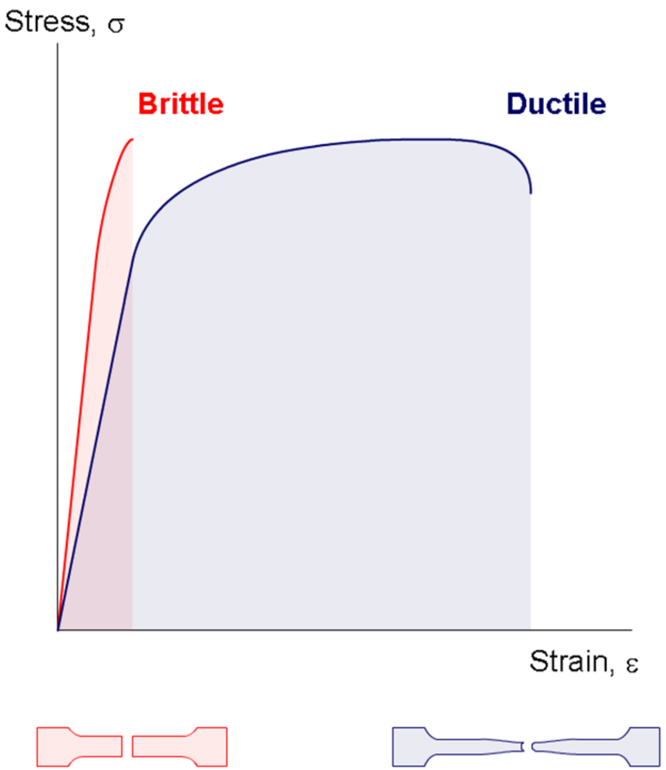
PURDUE ENGINEERING

Motivation

- We have spent the last few classes finding the **state of stress** at various points in a body due to **combined loading**
  - We have seen various combinations of normal stresses and shear stresses
- **Mohr's circle** gives us a way to **compare** different states of stress
  - For **any** state of stress, we can identify three important parameters: the **two in-plane principal stresses** and the **absolute maximum shear stress**
- Now: how can we **use** this information to predict whether a point in a body will **fail**?
  - First, we need to define what "failure" means…this depends on the type of material!

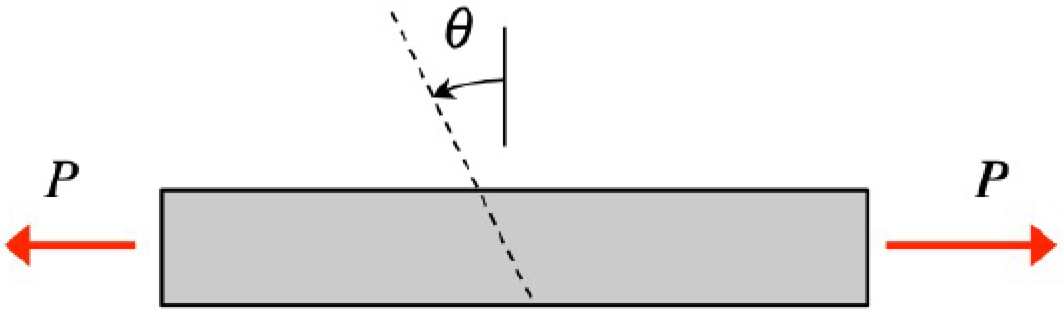
Failure theories overview

- Use the results of “simple” tests to formulate hypotheses
  - Usual hypothesis: the mechanism that causes failure in a tensile test is *the same mechanism* that causes failure in more complex stress states
- We have different failure theories for brittle and ductile materials
- “All models are wrong, but some are useful”

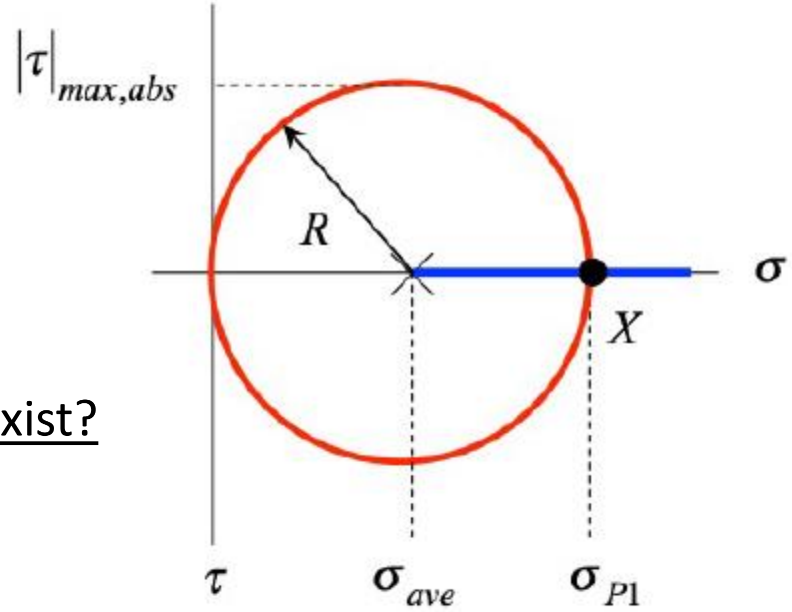


The tensile test Lecture Book: Ch. 15, pg. 2

Free-body diagram



Mohr's circle at any point on any cross section



What is the maximum normal stress? On which plane does this stress exist?

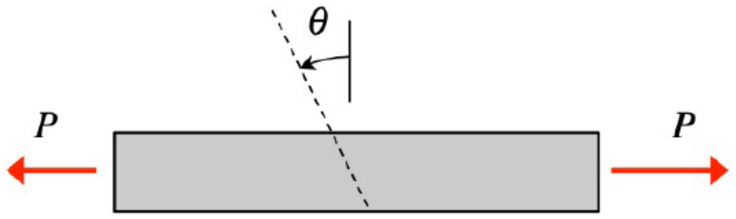
What is the maximum shear stress? On which planes does this stress exist?

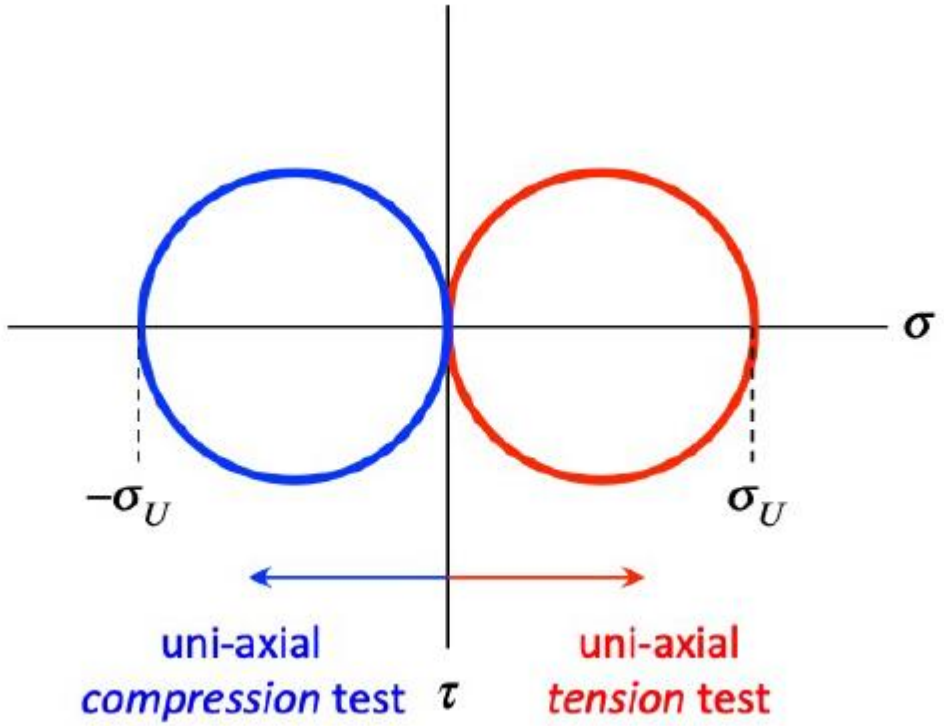
Brittle failure: Maximum normal stress theory

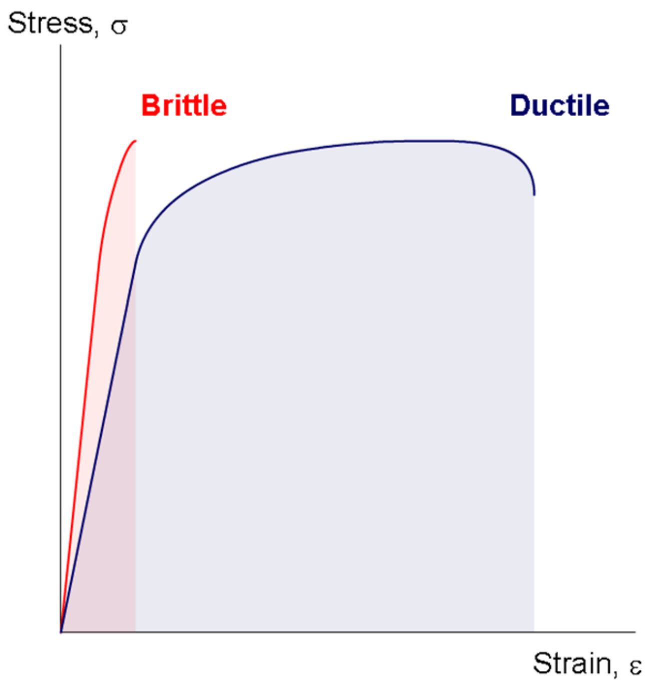
<u>Hypothesis</u>: A brittle material fractures when the maximum principal stress equals or exceeds the ultimate normal stress when fracture occurs in a tensile test

Define this “ultimate normal stress” as the **ultimate strength**

Assumption: the ultimate strength in tension and compression is the same



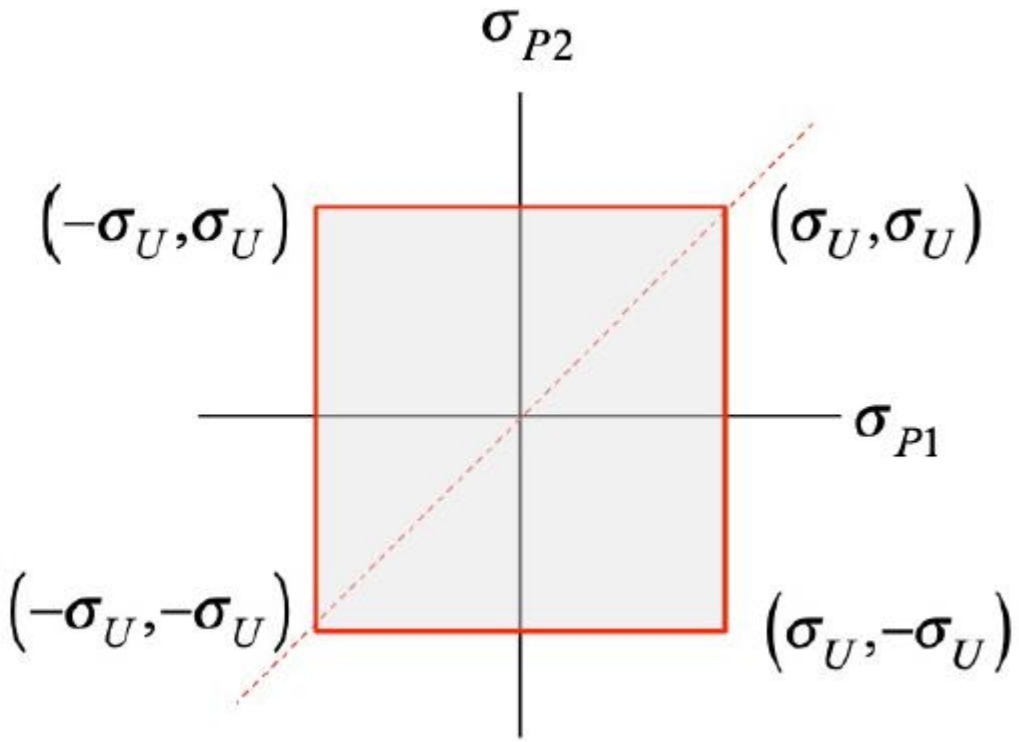




Lecture Book: Ch. 15, pg. 11

Brittle failure: Maximum normal stress theory

We can visualize the failure boundary in principal stress space



<u>Failure criteria</u>

$$\sigma_{p1} \geq \sigma_U$$

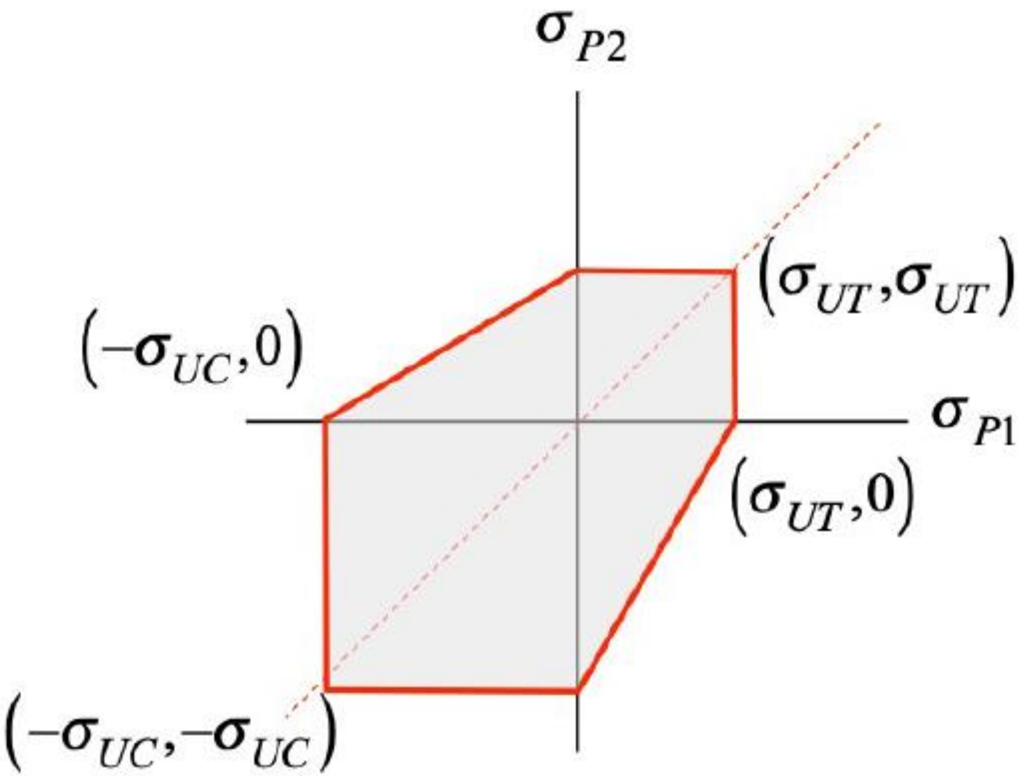
$$\sigma_{p2} \leq -\sigma_U$$

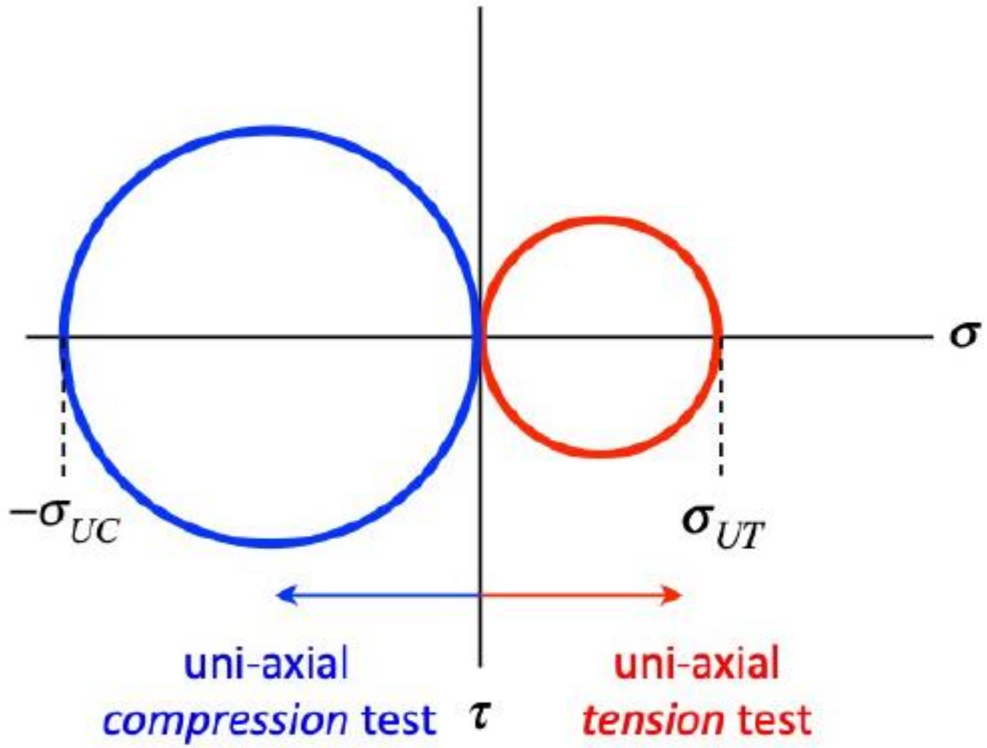
<u>Factor of safety</u>

Lecture Book: Ch. 15, pg. 11

Brittle failure: Mohr's theory

Modification to maximum normal stress theory based on the observation that many materials are <u>stronger in compression than they are in tension</u>, i.e. $\sigma_{UT} < \sigma_{UC}$, and the maximum normal stress theory is non-conservative when the principal stresses have different signs



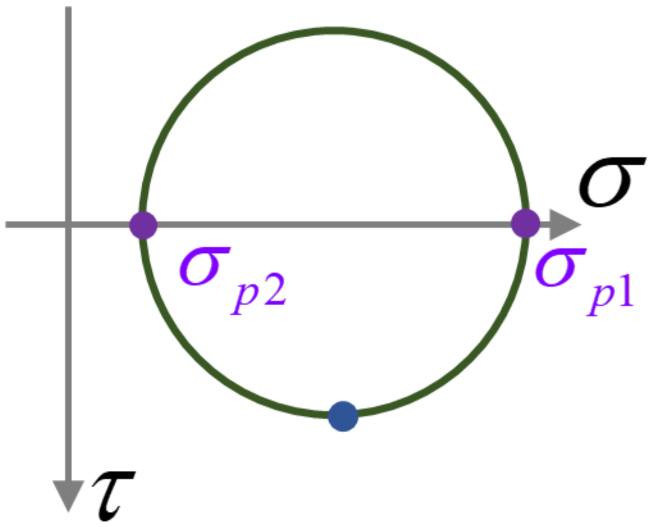


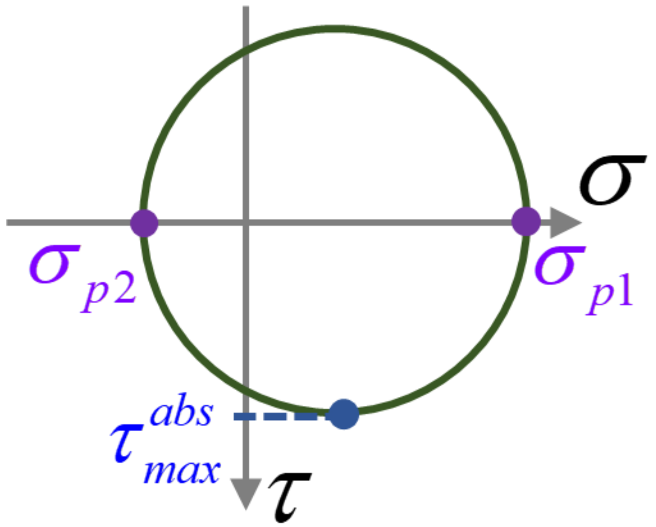
Lecture Book: Ch. 15, pg. 12

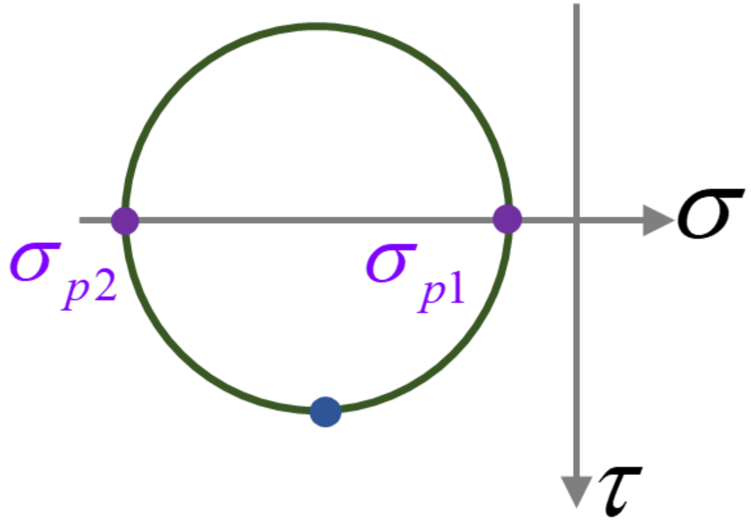
Brittle failure: Mohr's theory

For a general state of plane stress, there are three possible situations **Lecture Book: Ch. 15, pg. 6**

Case 1: $\sigma_{p1} > \sigma_{p2} > 0$ Case 2: $\sigma_{p1} > 0 > \sigma_{p2}$ Case 3: $0 > \sigma_{p1} > \sigma_{p2}$



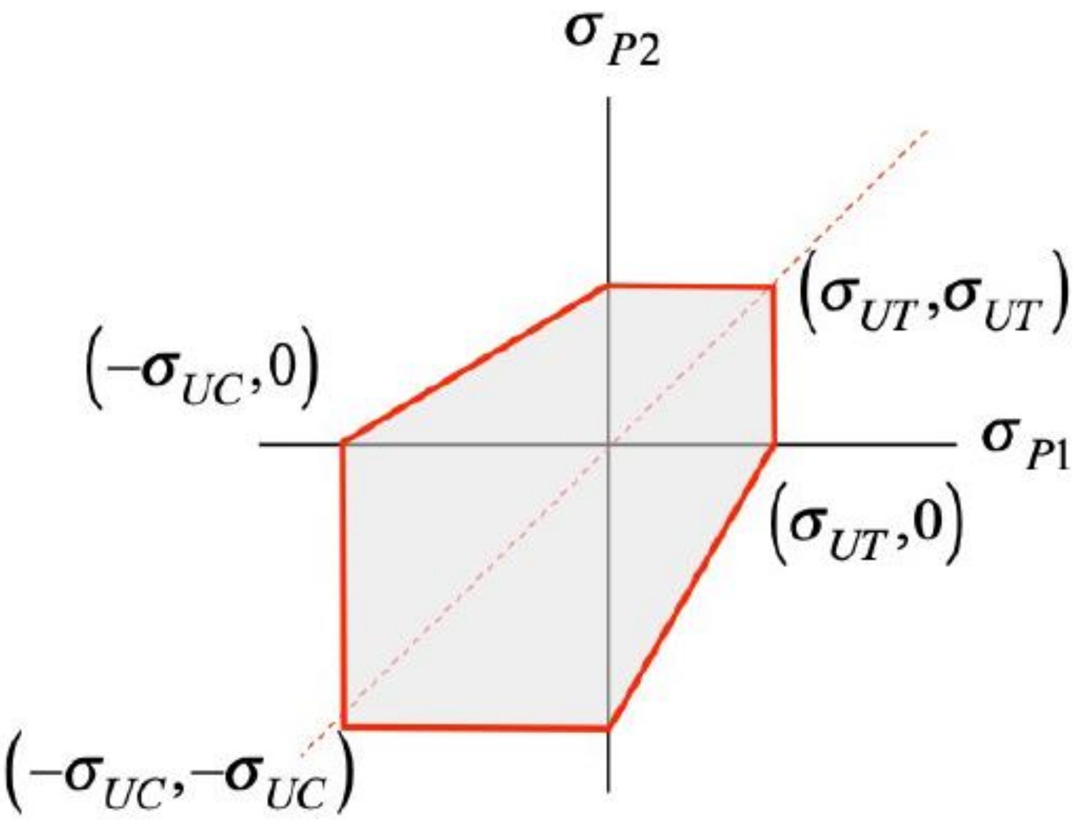




Lecture Book: Ch. 15, pg. 12

Brittle failure: Mohr's theory

We can visualize the failure boundary in principal stress space



Failure criteria

Case 1: $\sigma_{p1} > \sigma_{p2} > 0$

$$\sigma_{p1} \geq \sigma_{UT}$$

Case 2: $\sigma_{p1} > 0 > \sigma_{p2}$

$$\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} \geq 1$$

Case 3: $0 > \sigma_{p1} > \sigma_{p2}$

$$\sigma_{p2} \leq -\sigma_{UC}$$

Factor of safety

Lecture Book: Ch. 15, pg. 12

Brittle failure: Summary

Maximum normal stress theory

Failure criterion

$$\sigma_{p1} \geq \sigma_U \quad \text{or} \quad \sigma_{p2} \leq -\sigma_U$$

Factor of safety

$$FS = \left|\frac{\sigma_U}{\sigma_{p1}}\right| \quad \text{or} \quad FS = \left|\frac{\sigma_U}{\sigma_{p2}}\right|$$

(whichever is *smaller* is the real factor of safety)

Mohr's failure theory

Failure criteria (3 possible cases based on the signs of the principal stresses

$$\sigma_{p1} > \sigma_{p2} > 0: \; \sigma_{p1} \geq \sigma_{UT} \qquad 0 > \sigma_{p1} > \sigma_{p2}: \; \sigma_{p2} \leq -\sigma_{UC}$$

$$\sigma_{p1} > 0 > \sigma_{p2}: \; \frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}} \geq 1$$

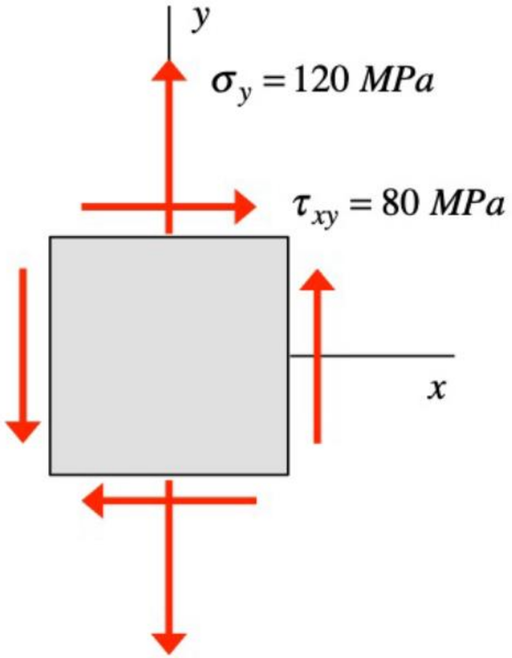
Factor of safety

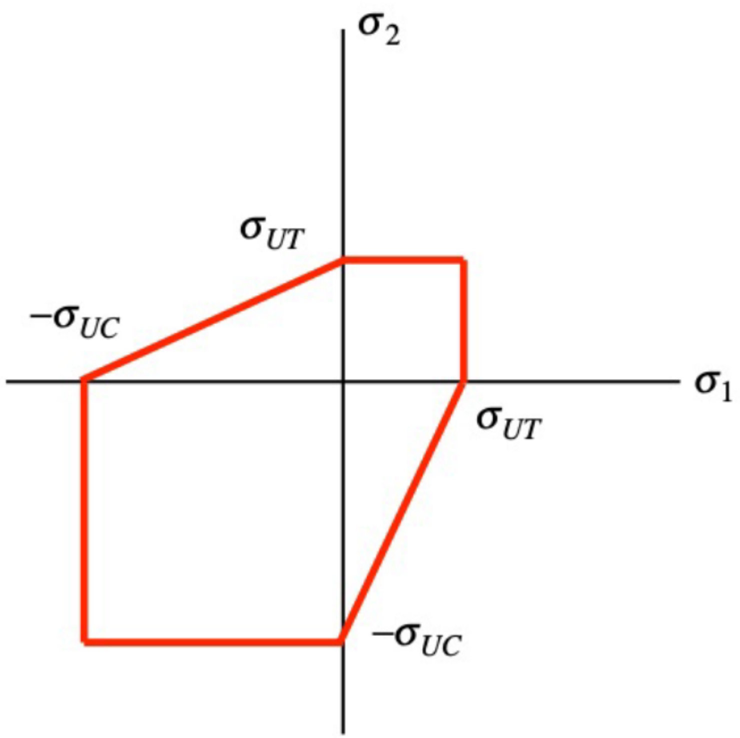
$$\sigma_{p1} > \sigma_{p2} > 0: \; FS = \frac{\sigma_{UT}}{\sigma_{p1}} \qquad 0 > \sigma_{p1} > \sigma_{p2}: \; FS = \left|\frac{\sigma_{UC}}{\sigma_{p2}}\right|$$

$$\sigma_{p1} > 0 > \sigma_{p2}: \; FS = \frac{1}{\frac{\sigma_{p1}}{\sigma_{UT}} - \frac{\sigma_{p2}}{\sigma_{UC}}} = \frac{\sigma_{UT}\sigma_{UC}}{\sigma_{p1}\sigma_{UC} - \sigma_{p2}\sigma_{UT}}$$

Example 15.7

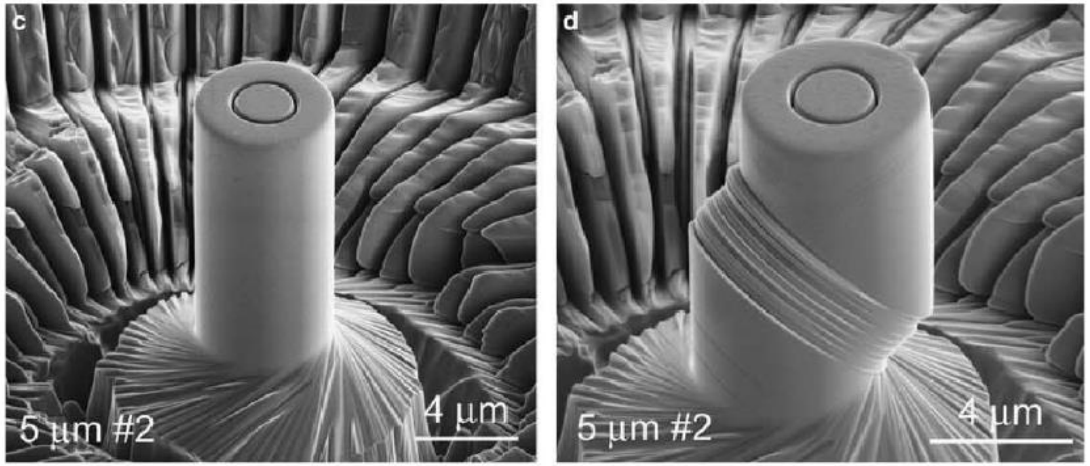
The state of stress shown exists at a location in a component made of a brittle material with $\sigma_{UC} = 850\ \text{MPa}$ and $\sigma_{UT} = 170\ \text{MPa}$. According to Mohr's theory, has the material failed?





Ductile failure: Maximum shear stress theory

- On the microscale, permanent (plastic) deformation occurs by “slip”



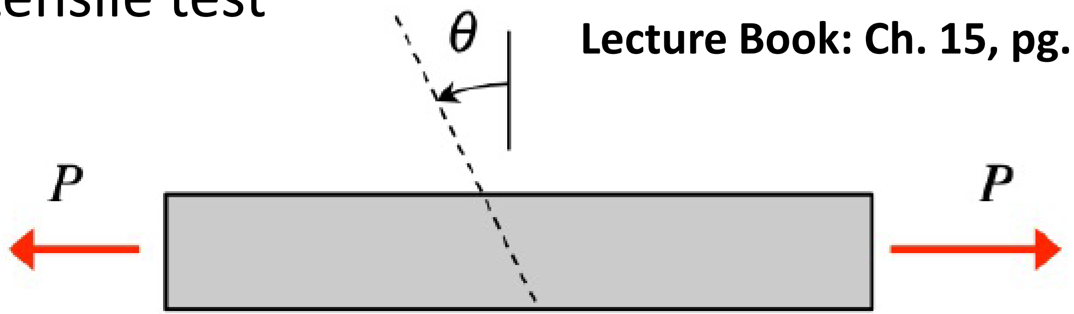
- Failure in a tensile test of a ductile material often looks very similar

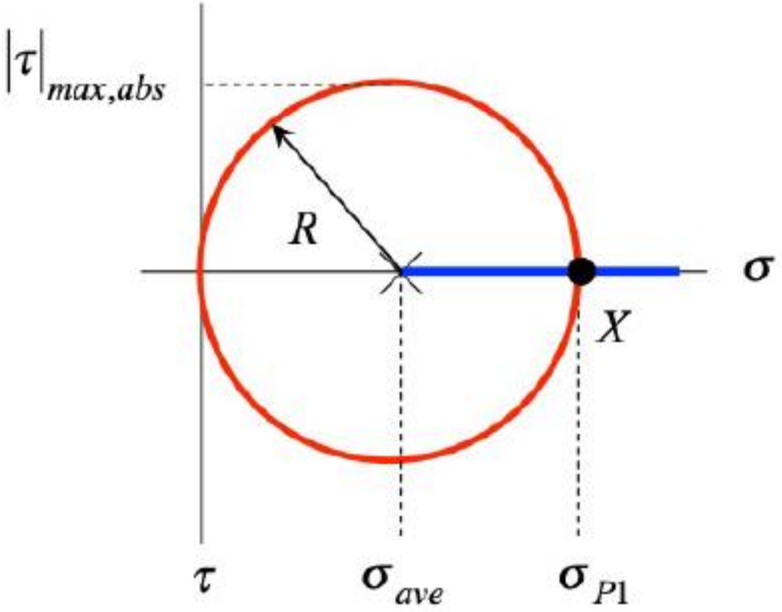
Aluminum – failure due to normal or shear stress?

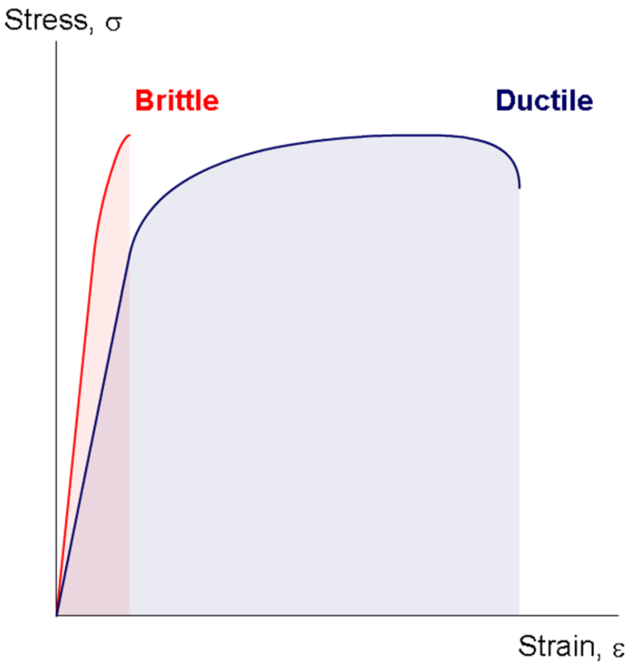
Ductile failure: Maximum shear stress theory

Hypothesis: for *any* stress state, yielding of a ductile material occurs when the *absolute maximum shear stress* equals or exceeds the maximum shear stress when yielding occurs in a tensile test

Lecture Book: Ch. 15, pg. 5





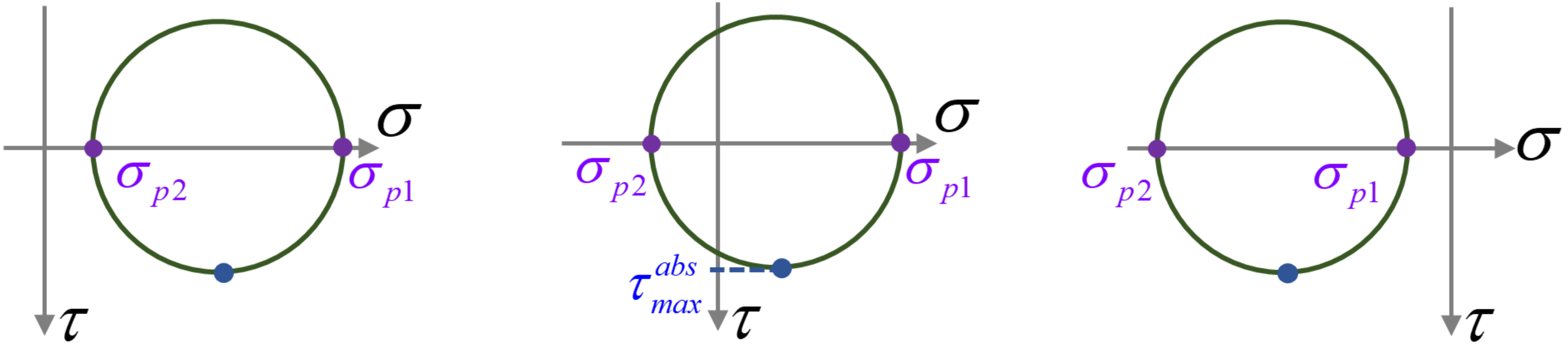


Lecture Book: Ch. 15, pg. 5

Ductile failure: Maximum shear stress theory

For a general state of plane stress, there are three possible situations **Lecture Book: Ch. 15, pg. 6**

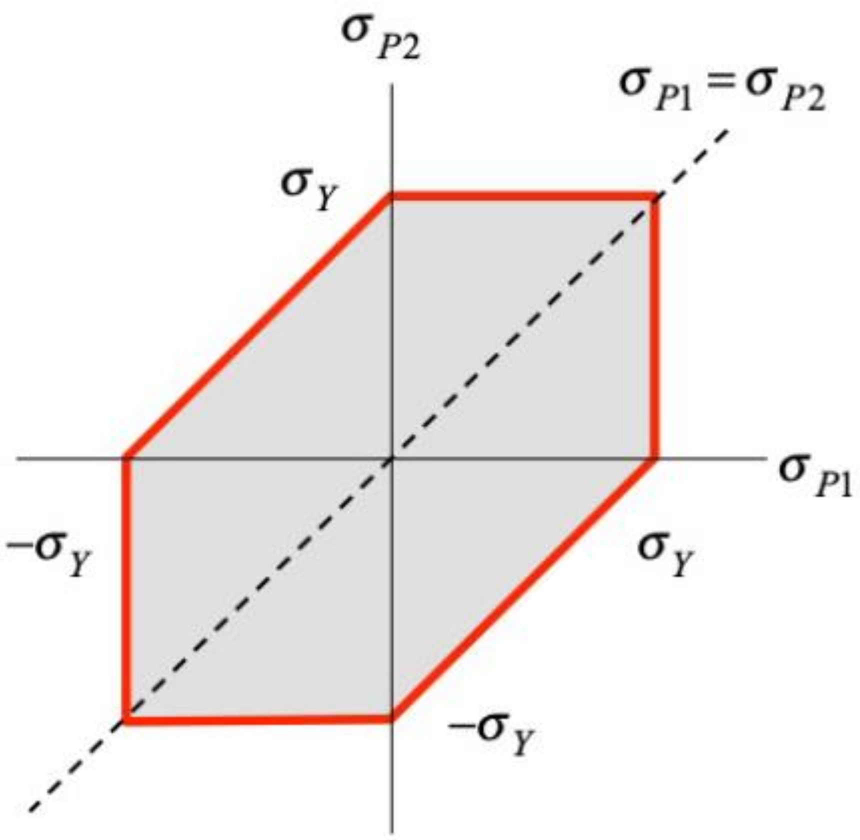
<u>Case 1:</u> $\sigma_{p1} > \sigma_{p2} > 0$ <u>Case 2:</u> $\sigma_{p1} > 0 > \sigma_{p2}$ <u>Case 3:</u> $0 > \sigma_{p1} > \sigma_{p2}$



Ductile failure: Maximum shear stress theory

We can visualize the failure boundary in principal stress space

Lecture Book: Ch. 15, pg. 7



Failure criteria

Case 1: $\sigma_{p1} > \sigma_{p2} > 0$

$$\sigma_{p1} \geq \sigma_Y$$

Case 2: $\sigma_{p1} > 0 > \sigma_{p2}$

$$\sigma_{p1} - \sigma_{p2} \geq \sigma_Y$$

Case 3: $0 > \sigma_{p1} > \sigma_{p2}$

$$|\sigma_{p2}| \geq \sigma_Y$$

Factor of safety

Ductile failure: Maximum distortional energy theory

von Mises proposed a <u>different hypothesis</u>: yielding occurs when the *distortion energy density* equals or exceeds the distortion energy density when yielding occurs in a tensile test

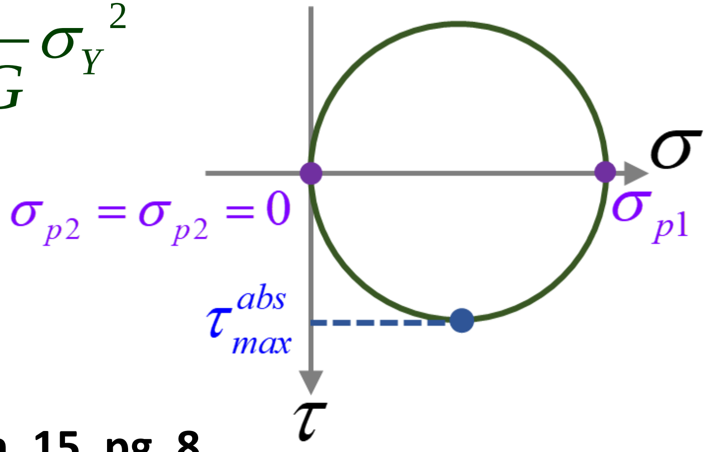
Evidence: a material subjected to purely hydrostatic stress $(\sigma_{p1} = \sigma_{p2} = \sigma_{p3})$ *never* yields

Total elastic strain energy density = change of volume + distortion (change of shape)

$$\bar{u} = \frac{1}{2E}\left[\sigma_{p1}^2 + \sigma_{p2}^2 - 2\nu\sigma_{p1}\sigma_{p2}\right] \qquad \bar{u}_v = \frac{1}{2G}\left(\sigma_{p1} + \sigma_{p2}\right) \qquad \bar{u}_d = \frac{1}{6G}\left(\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right)$$

<u>For yielding in the tensile test</u>

$$\bar{u}_{d,yield} = \frac{1}{6G}\sigma_Y^{\ 2}$$



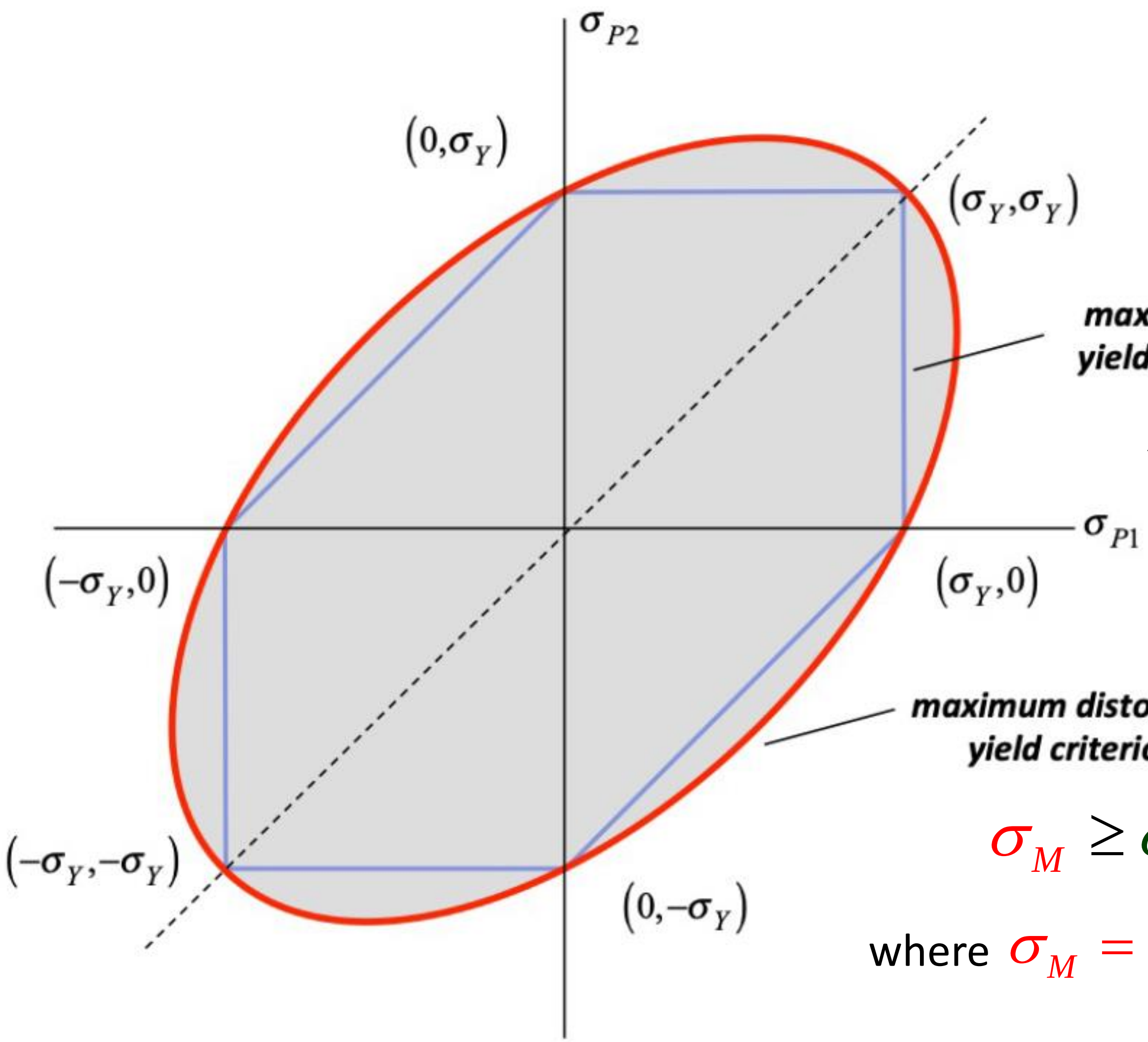
So, our failure criterion for *<u>any plane stress state</u>* is

$$\frac{1}{6G}\boxed{\left(\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2\right)} = \frac{1}{6G}\boxed{\sigma_Y^{\ 2}}$$

Lecture Book: Ch. 15, pg. 8

Ductile failure: Maximum distortional energy theory

In principal stress space, the maximum distortional energy failure boundary is an ellipse



$$\tau_{abs}^{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_Y}{2}$$

Failure criterion

$$\sigma_M \geq \sigma_Y$$

where $\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2}$

Factor of safety

Ductile failure: Summary

Maximum shear stress theory

Failure criterion: $\tau_{max}^{abs} \geq \frac{\sigma_Y}{2}$

3 possible cases for τ_{max}^{abs} based on signs of principal stresses

$\sigma_{p1} > \sigma_{p2} > 0: \ \sigma_{p1} \geq \sigma_Y \qquad 0 > \sigma_{p1} > \sigma_{p2}: \ |\sigma_{p2}| \geq \sigma_Y$

$\sigma_{p1} > 0 > \sigma_{p2}: \ \sigma_{p1} - \sigma_{p2} \geq \sigma_Y$

Or, if you re-order the principal stresses so $\sigma_1 > \sigma_2 > \sigma_3$,

$\tau_{abs}^{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{\sigma_Y}{2}$ is the failure criterion for all cases

Factor of safety: $FS = \frac{\sigma_Y}{2\tau_{max}^{abs}} = \frac{\sigma_Y}{\sigma_1 - \sigma_3}$

Maximum distortional energy (von Mises) theory

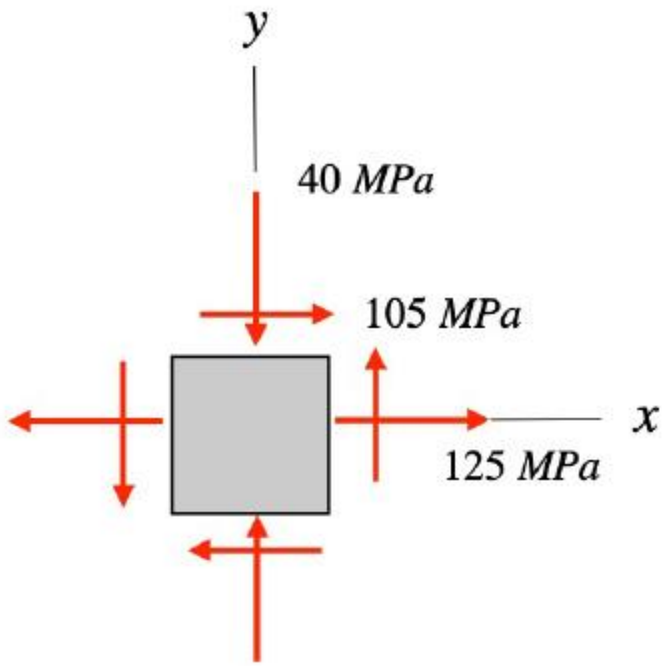
Failure criterion (based on the von Mises stress):

$$\sigma_M = \sqrt{\sigma_{p1}^2 - \sigma_{p1}\sigma_{p2} + \sigma_{p2}^2} \geq \sigma_Y$$

Factor of safety: $FS = \frac{\sigma_Y}{\sigma_M}$

Example 15.1

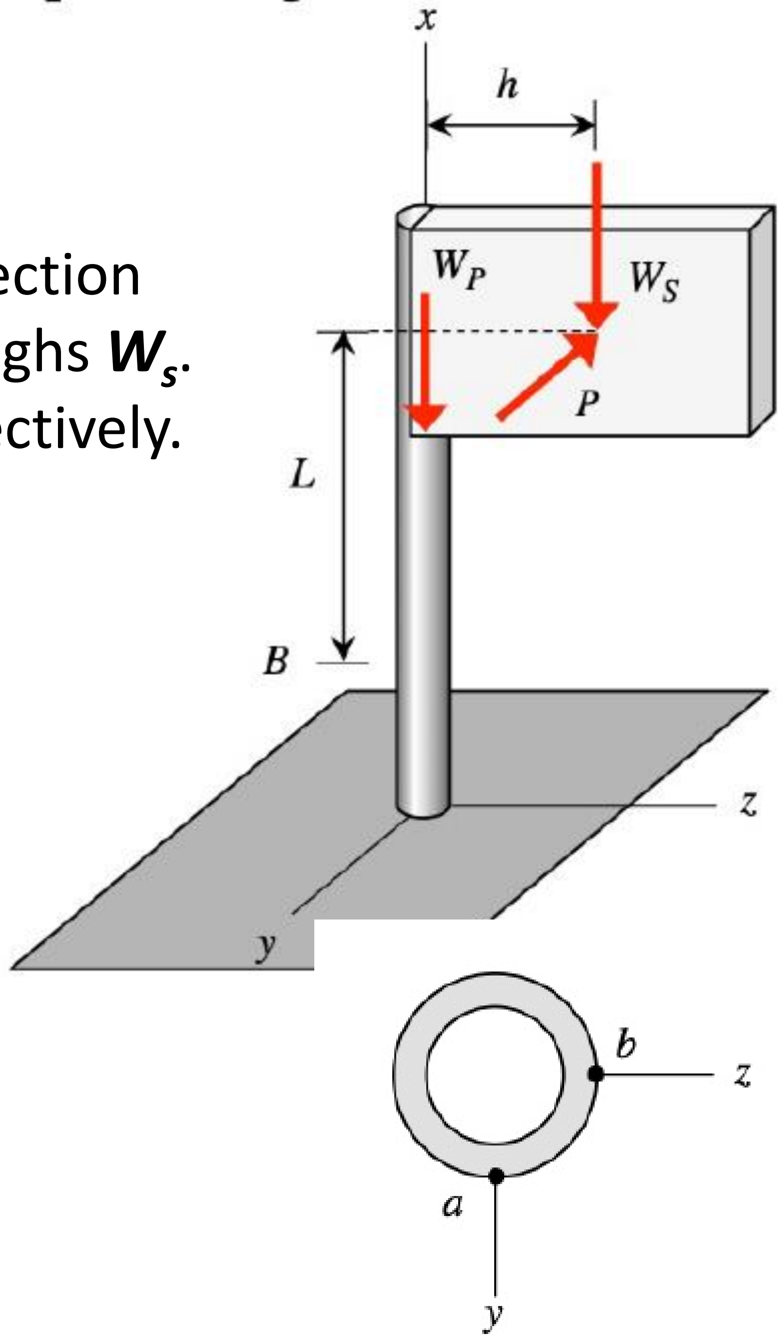
The state of stress shown is in a component made of a ductile material with a **yield strength** of $\sigma_Y = 250\ \mathrm{MPa}$. Does the maximum shear stress theory predict failure for the material? Does the maximum distortion energy predict failure for the material?



Revisit Example 14.12

Wind blowing on a sign produces a resultant force ***P*** in the –y direction at the point shown. The support pole weighs $\boldsymbol{W_P}$ and the sign weighs $\boldsymbol{W_s}$. The pole is a pipe with outer and inner diameters $\boldsymbol{d_o}$ and $\boldsymbol{d_i}$, respectively.

What are the factors of safety for points *a* and *b* according to the maximum distortion energy theory if the pole is made from an aluminum alloy with a yield strength of 20 ksi?



pipe cross section at B

Bonus example

Determine the principal stresses and the maximum shear stress at point A (i.e., the point on top of the wrench handle). The diameter of the circular cross section is 12.5 mm.

If the wrench is made of a ductile material with a yield strength of 300 MPa, what value of the force will cause yielding at point A according to the maximum shear stress theory? How about the maximum distortion energy theory?