Lecture 42: Failure analysis – Buckling of columns

failure from axial load

if $|\sigma| = |P|/A \geq \sigma_Y$

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Lecture Book: Ch. 18

Final Exam

Monday, Dec. 9

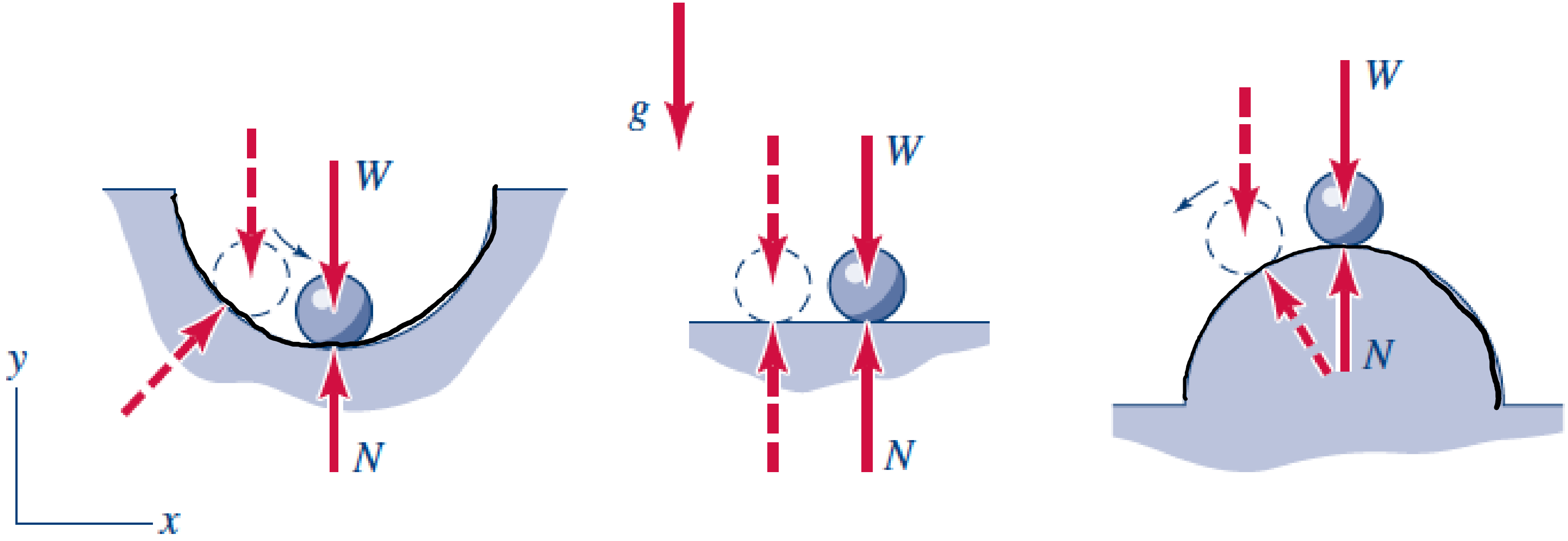
3:30 - 5:30 PM

WTHR 172

PURDUE ENGINEERING

Stability and equilibrium

What happens if we are in a state of unstable equilibrium?



Stable

Neutral

Unstable

Buckling experiment

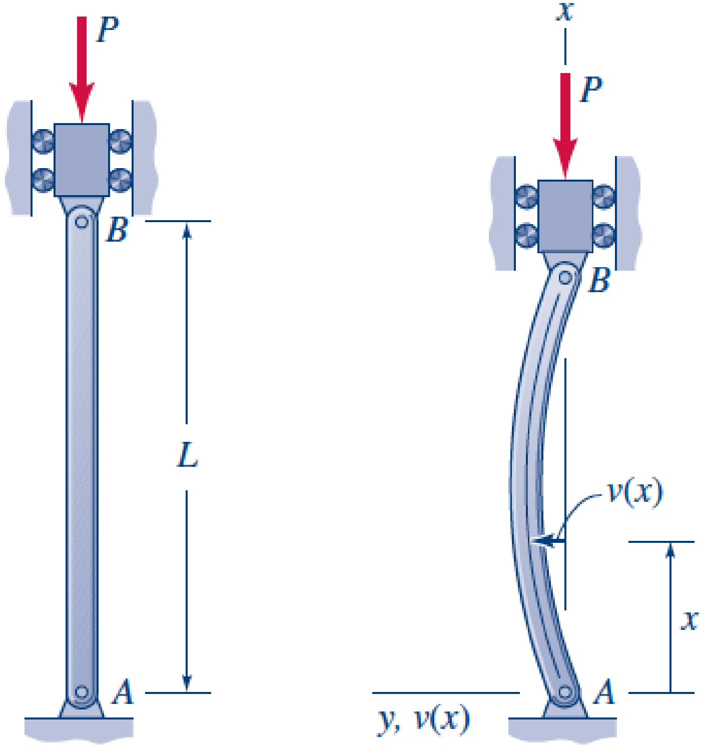
There is a critical stress at which *buckling* occurs depending on the material and the geometry

How do the material properties and geometric parameters influence the buckling stress?

Decrease L → higher critical buckling stress

Larger E → " " " "
(stiffer material)

BCs are important



Euler buckling equation

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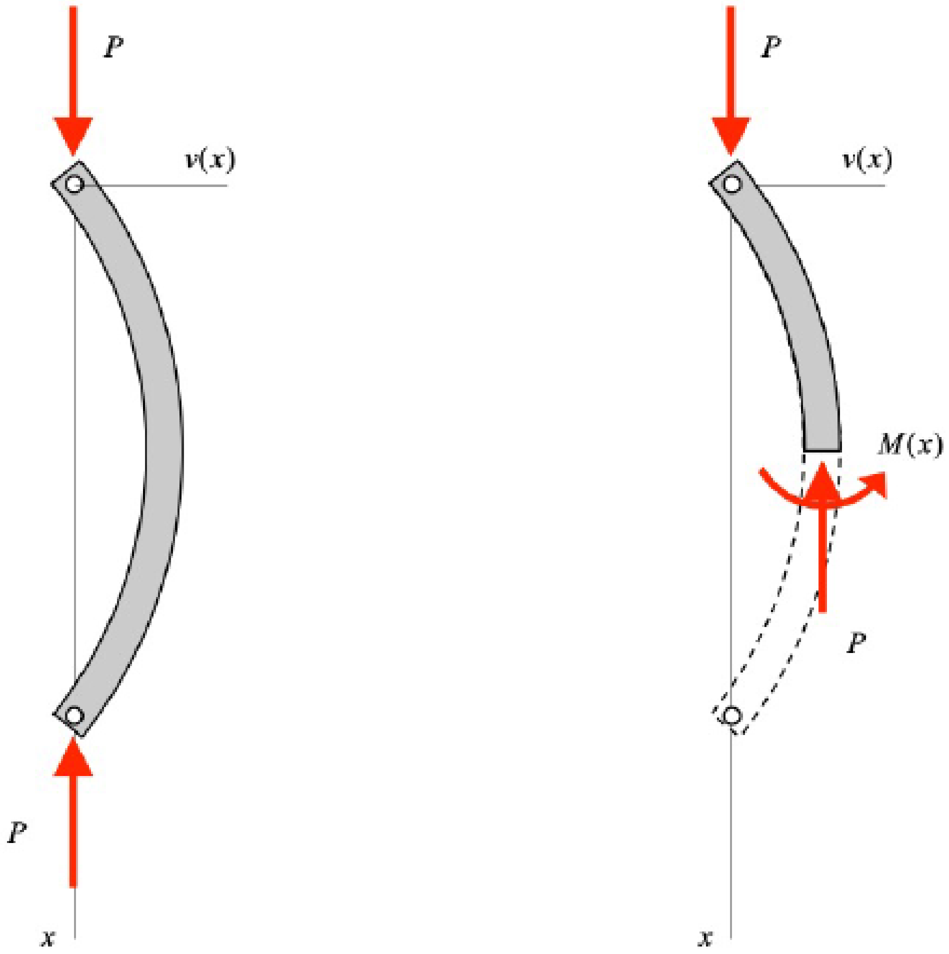
Consider static equilibrium of the *buckled* pinned-pinned column



Equil: $\sum M_{cut} = M(x) + Pv(x) = 0$

Moment-curvature: $M(x) = EI \, d^2v/dx^2$

$$\Rightarrow EI \frac{d^2v}{dx^2} + Pv(x) = 0$$

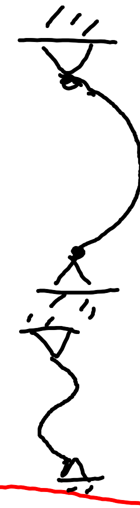
$$\left. \begin{array}{l} v(0) = 0 \\ v(L) = 0 \end{array} \right] \rightarrow BCs$$

Euler buckling equation

We have a differential equation for the deflection with BCs at the pins:

$$EI\frac{d^2v}{d^2x}+Pv(x)=0 \qquad \underline{v(0)=0} \quad \text{and} \quad \underline{v(L)=0}$$

$\boxed{n=1 \rightarrow}$



n=2

The solution is:

$$v(x)=\cancel{A}\overset{\to 0}{}\cos\left(x\sqrt{\frac{P}{EI}}\right)+B\sin\left(x\sqrt{\frac{P}{EI}}\right) \qquad \text{with}$$

$$A=0$$

$$B\sin\left(L\sqrt{\frac{P}{EI}}\right)=0 \quad \Rightarrow \quad \boxed{L\sqrt{\frac{P}{EI}}=n\pi,\quad n=1,2,3,\ldots}$$

$B=0 \rightarrow v(x)=0$

$\sin\left(L\sqrt{P/EI}\right)=0$

$$P=\frac{n^2\pi^2EI}{L^2}$$

Critical load for buckling : $\boxed{P_{cr}=\pi^2\frac{EI}{L_e^2}}$

Replace "L" with an effective length "L_e" for different BCs

Effect of boundary conditions

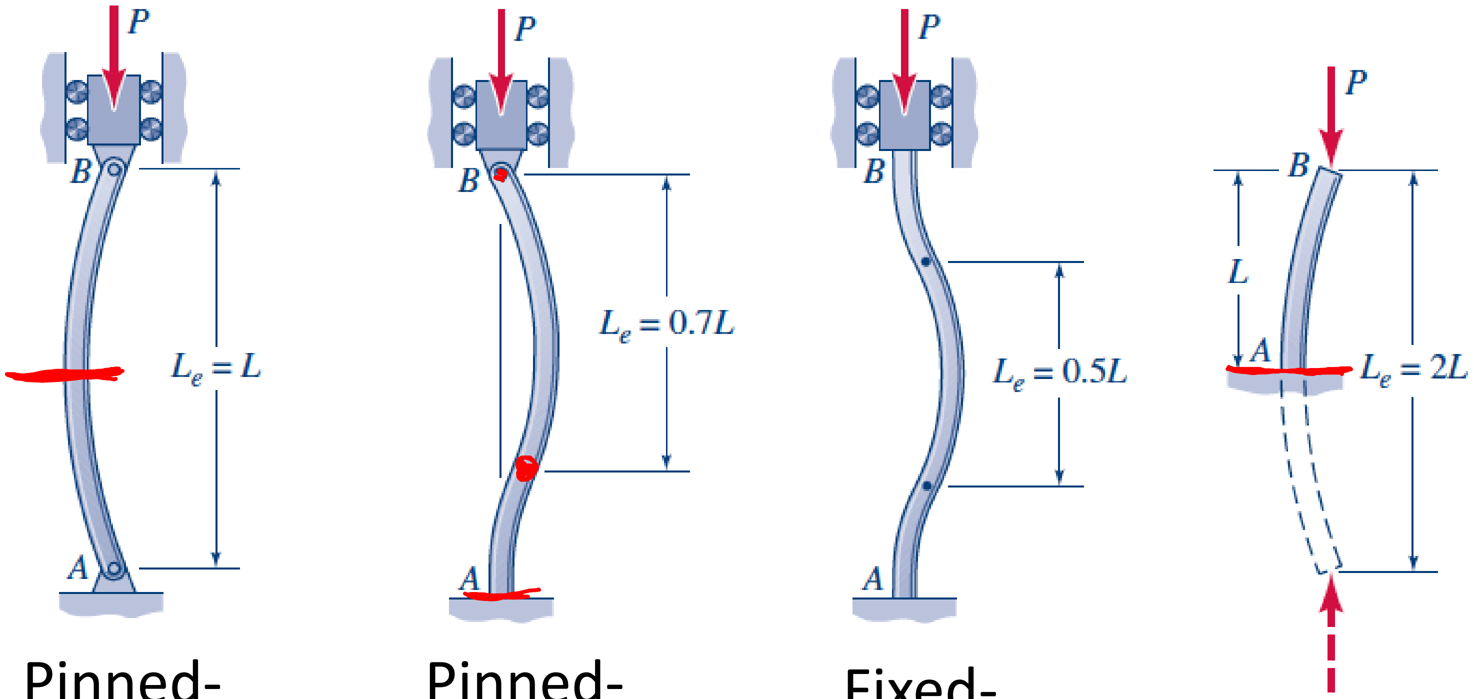
Critical load and critical stress for buckling:

$$\boxed{P_{cr} = \pi^2 \frac{EI}{L_e^2}} = \pi^2 \frac{EA}{\left(L_e/r_g\right)^2}$$

$$\boxed{\sigma_{cr} = \pi^2 \frac{E}{\left(L_e/r_g\right)^2}}$$

where $r_g = \sqrt{\frac{I}{A}}$ → units of length

is the “radius of gyration”



Pinned-pinned	Pinned-fixed	Fixed-fixed	Fixed-free
$L_e = L$	$L_e = 0.7L$	$L_e = 0.5L$	$L_e = 2L$

Think of σ_{cr} as a strength

Modifications to Euler buckling theory

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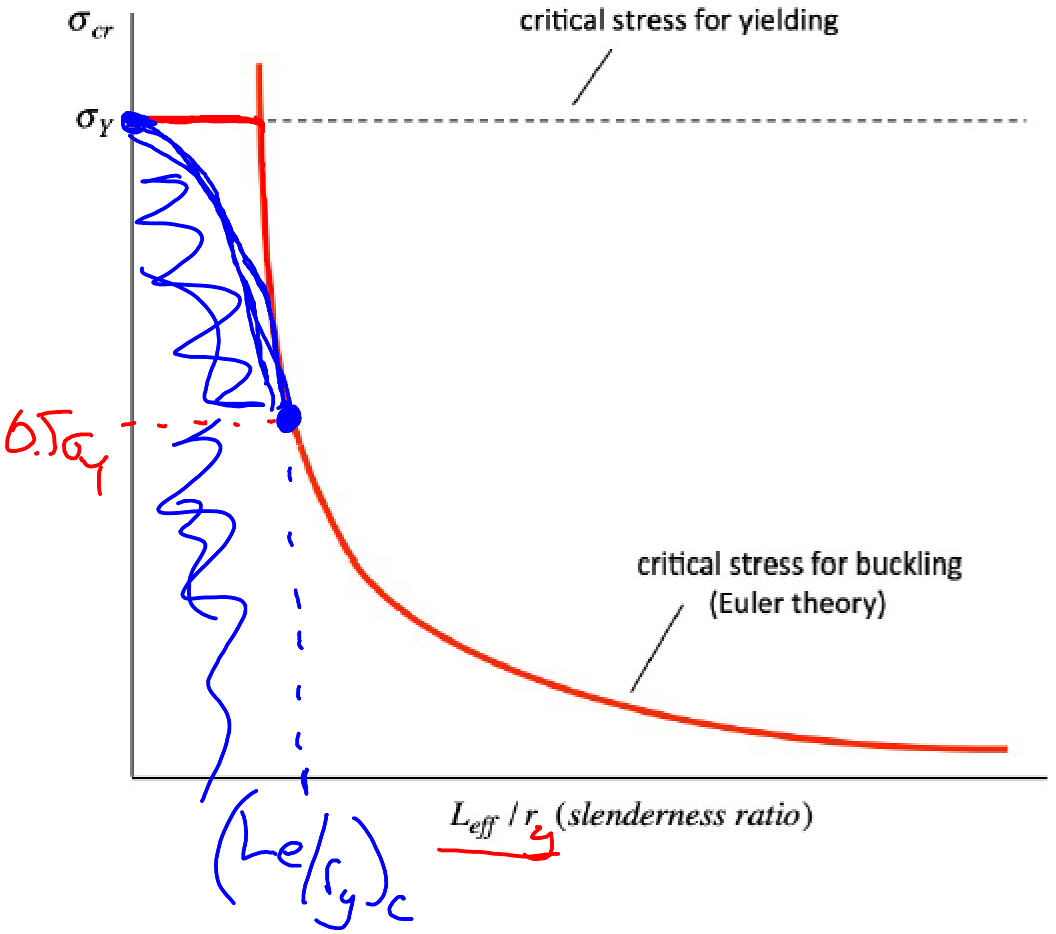
Euler buckling equation: works well for slender rods

long + thin

L_e/r_g

Needs to be modified for smaller "slenderness ratios" (where the critical stress for Euler buckling is at least half the yield strength)

$$\left(L_e/r_g\right)_c = \sqrt{\frac{\pi^2 E}{0.5\sigma_Y}}$$



If $\left(L_e/r_g\right) < \left(L_e/r_g\right)_c$, then use "Johnson's buckling equation"

Summary

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Critical slenderness ratio: $\left(\frac{L_e}{r_g}\right)_c = \sqrt{\frac{\pi^2 E}{0.5\sigma_Y}}$

Euler buckling (high slenderness ratio):

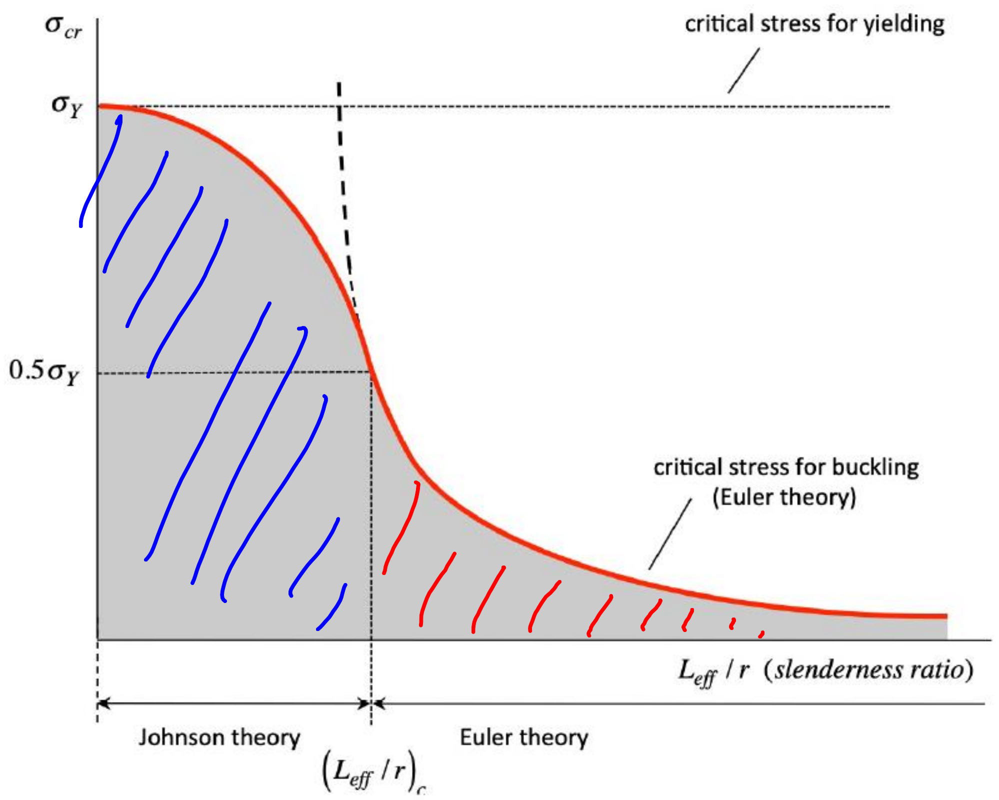
$\sigma_{cr} = \frac{P_{cr}}{A}$

If $\left(\frac{L_e}{r_g}\right) > \left(\frac{L_e}{r_g}\right)_c$: $\sigma_{cr} = \frac{\pi^2 E}{\left(L_e/r_g\right)^2}$ or $\boxed{P_{cr} = \pi^2 \frac{EI}{L_e^2}}$

Johnson buckling (low slenderness ratio):

If $\left(\frac{L_e}{r_g}\right) < \left(\frac{L_e}{r_g}\right)_c$: $\sigma_{cr} = \left[1 - \frac{\left(L_e/r_g\right)^2}{2\left(L_e/r_g\right)_c^2}\right]\sigma_Y$

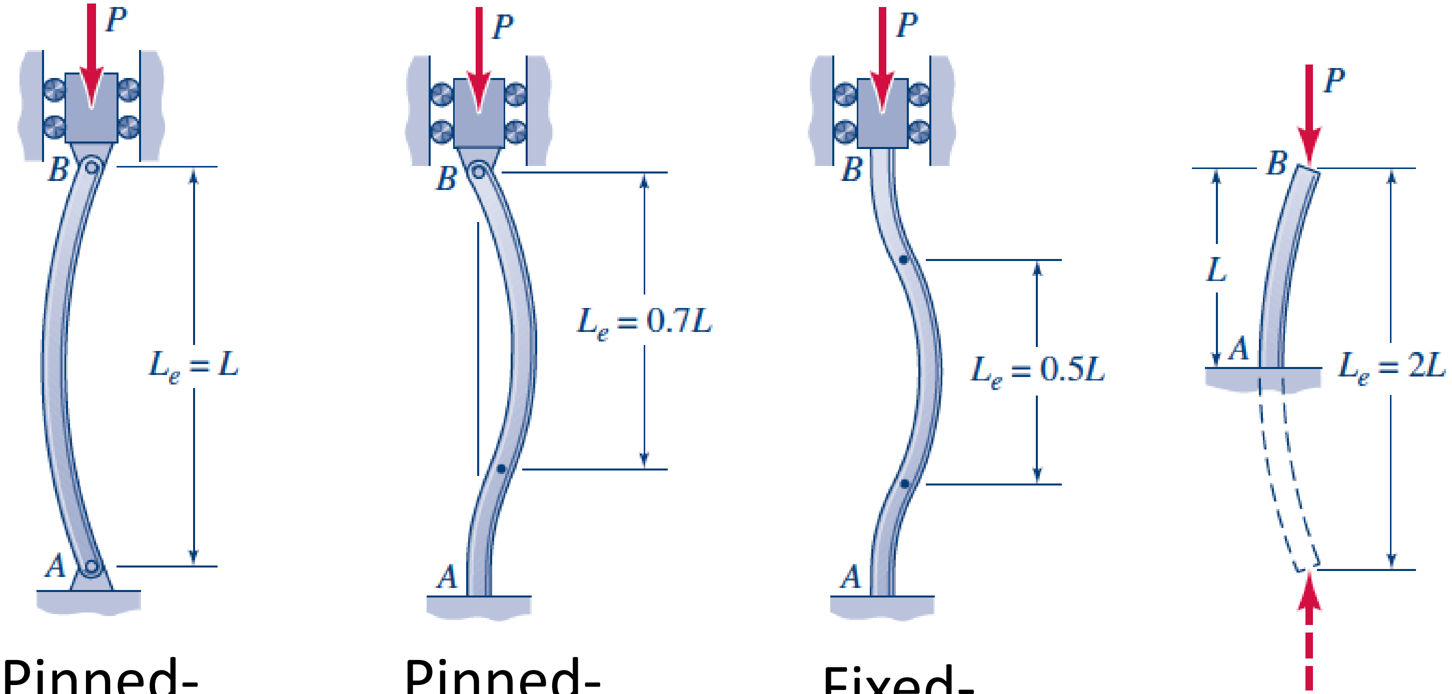
with radius of gyration $r_g = \sqrt{\frac{I}{A}}$



Summary

Buckling only occurs in compression

Effective length from the boundary conditions:



Pinned-pinned	Pinned-fixed	Fixed-fixed	Fixed-free
$L_e = L$	$L_e = 0.7L$	$L_e = 0.5L$	$L_e = 2L$

Example 18.1

Determine the critical buckling load P_{cr} of a steel pipe column that has a length of L with a tubular cross section of inner radius r_i and thickness t. The material has Young's modulus E and yield strength σ_Y. Use pinned-fixed boundary conditions.

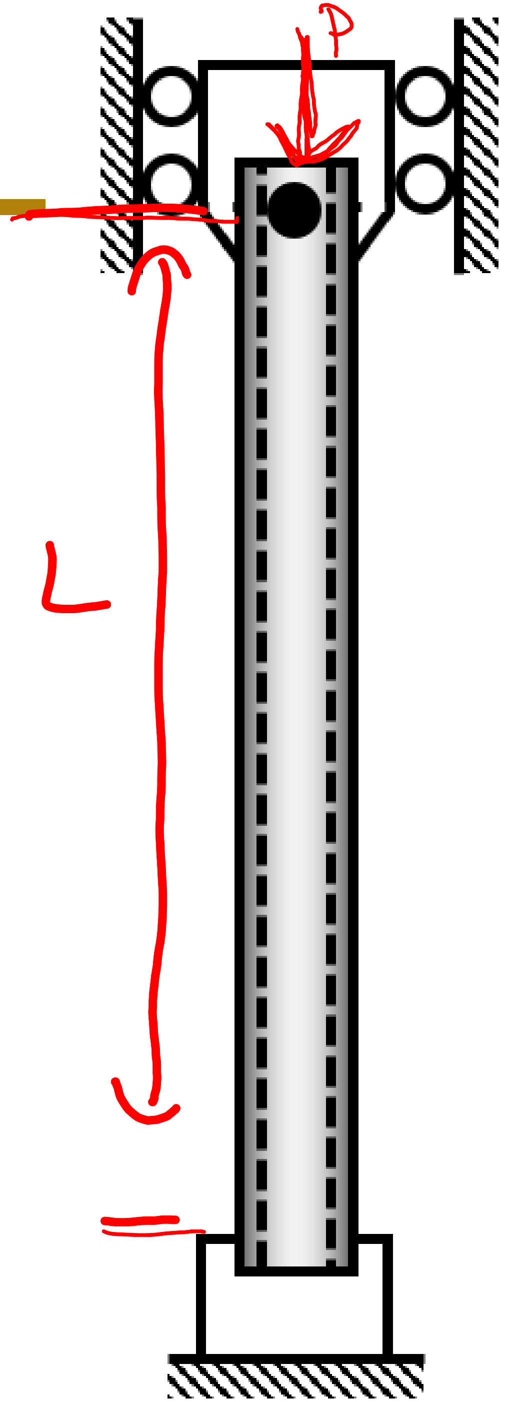
Use Euler's buckling theory

$$P_{cr} = \pi^2 \frac{EI}{L_e^2}$$

$$I = \frac{\pi}{4}\left[(r_i+t)^4 - r_i^4\right]$$

$$L_e = 0.7L \quad \text{(pinned-fixed)}$$

$$\boxed{P_{cr} = \pi^2 \frac{E\left\{\frac{\pi}{4}\left[(r_i+t)^4 - r_i^4\right]\right\}}{(0.7L)^2}}$$



What happens to P_{cr} if we...

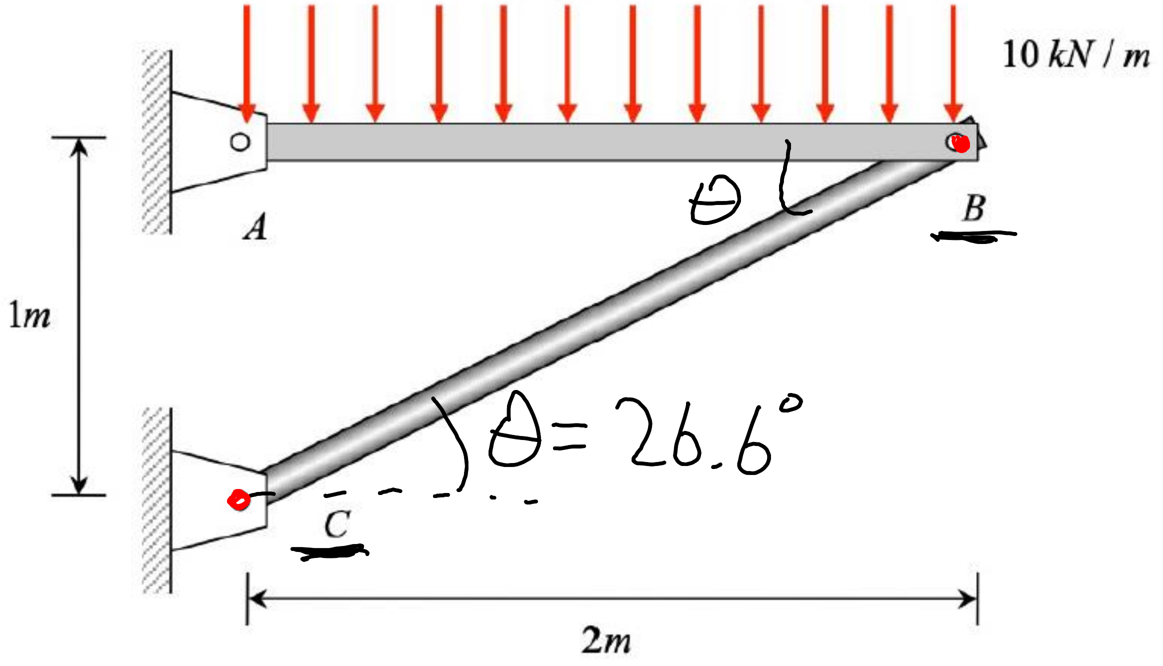
* Switch to pinned-pinned BC? $\rightarrow L_e = L \Rightarrow P_{cr}$ decreases
* Switch to fixed-fixed BC? $\rightarrow L_e = 0.5L < 0.7L \Rightarrow P_{cr}$ increases
* Increase the thickness t? $\rightarrow I$ increases $\Rightarrow P_{cr}$ increases

Example 18.2

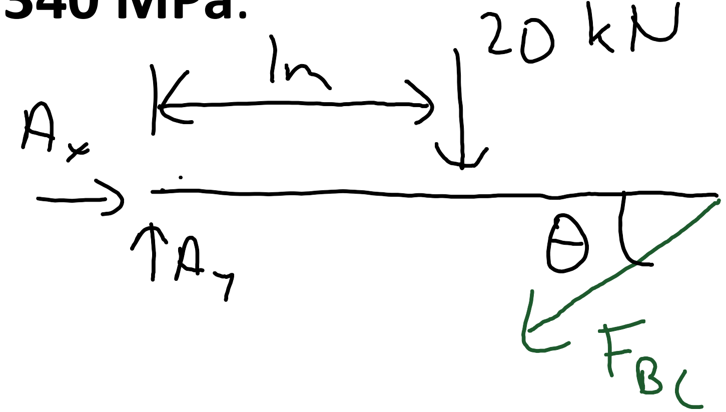
$d = 48\ mm$

$d_i = d - 2t = 38mm$

The steel compression strut BC of the frame ABC is a tube with an outer diameter of ***d* = 48 mm** and a wall thickness of ***t* = 5 mm**. Determine the factor of safety against elastic buckling if a distributed load of **10 kN/m** is applied to AB. Let ***E* = 210 GPa** and **σ_Y = 340 MPa**.



FBD + Equil



$$+\circlearrowleft \sum M_A = 0 = -(20kN)(1m) - (F_{BC} \sin\theta)(2m)$$

$$\Rightarrow F_{BC} = -22.4\ kN = -22\,400\ N$$

BC is in compression → buckling?

1.) Euler or Johnson

$$\left(\frac{L_e}{r_y}\right)_c = \sqrt{\frac{\pi^2 E}{0.5\sigma_y}} = \sqrt{\frac{\pi^2 (210\times10^3 \text{ MPa})}{0.5(340 \text{ MPa})}} = \boxed{110.4 = \left(\frac{L_e}{r_y}\right)_c}$$

$$\left(\frac{L_e}{r_y}\right) = ?$$

pinned - pinned $\rightarrow L_e = L_{BC} = 2.236\text{m} = \boxed{2236\text{ mm} = L_e}$

$$r_y = \sqrt{I/A} \leftarrow \begin{cases} I = \frac{\pi}{64}(d^4 - d_i^4) = 1.582\times10^5 \text{ mm}^4 \\ A = \frac{\pi}{4}(d^2 - d_i^2) = 675.4 \text{ mm}^2 \end{cases}$$

$$\Rightarrow \boxed{r_y = 15.3\text{ mm}}$$

$$\Rightarrow \left(\frac{L_e}{r_y}\right) = 146.1 > \left(\frac{L_e}{r_y}\right)_c = 110.4 \Rightarrow \boxed{\text{use Euler}}$$

2.) Find P_{cr} or σ_{cr}

Euler $\rightarrow$ $\sigma_{cr} = \dfrac{P_{cr}}{A} = \dfrac{\pi^2 E}{(L_e/r_y)^2} = 97.1\ \text{MPa} = \sigma_{cr}$

OR

$P_{cr} = \dfrac{\pi^2 E I}{L_e^2} = 6.56 \times 10^4\ \text{N} = 65.6\ \text{kN} = \sigma_{cr} A$

3.) Compare with the actual axial load or stress carried by the column

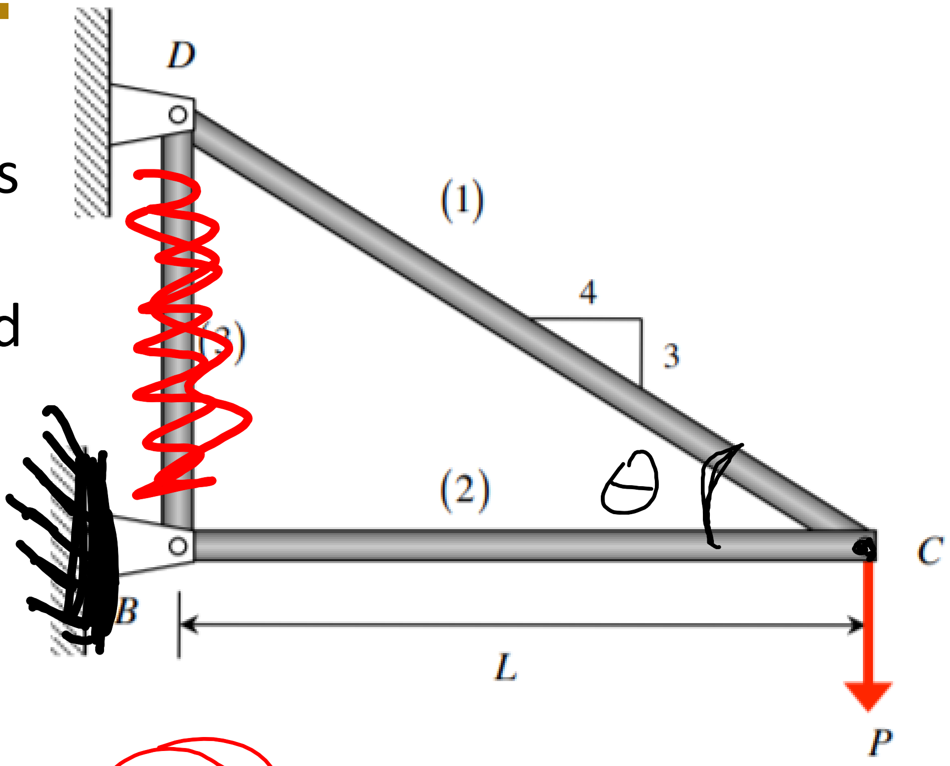
$|F_{BC}| = 22\,400\ \text{N}$ (compressive)

$\Rightarrow |\sigma_{BC}| = |F_{BC}|/A = 33.2\ \text{MPa}$

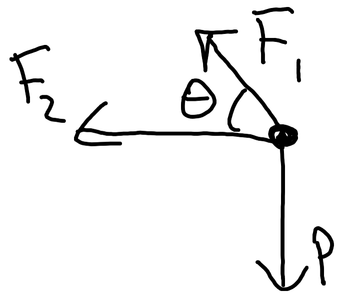
$$\boxed{FS = \frac{P_{cr}}{|F_{BC}|} = \frac{\sigma_{cr}}{\sigma_{BC}} = 2.9}$$

Example 18.8 (additional examples)

Members (1), (2), and (3) have Young's modulus $\boldsymbol{E = 10^7}$ **psi** and $\boldsymbol{\sigma_Y = 60 \times 10^3}$ **psi**. Each member has a solid circular cross section with diameter $\boldsymbol{d = 1}$ **in.** A force $\boldsymbol{P = 10}$ **kips** is applied to joint C. Determine the maximum length $\boldsymbol{L}$ that can be used without buckling based on **Euler's theory of buckling**.



FBD + Equil



$\cos\theta = 4/5 \quad \sin\theta = 3/5$

$$F_1 = +\frac{5}{3}P$$

$$\boxed{F_2 = \frac{-4}{3}P = \frac{-40}{3}\text{ kips}}$$

1.) Euler or Johnson

2.) Find σ_{cr} or P_{cr}

Euler $\rightarrow$ $P_{cr} = \pi^2 \frac{EI}{L_e^2}$

$E = 10^7$ psi

$I = \frac{\pi}{64} d^4 = \frac{\pi}{64}$ in^4

pinned-pinned: $L_e = L$

$$L = \sqrt{\frac{\pi^2 EI}{P_{cr}}}$$

3.) Compare with actual axial force

$P_{cr} = |F_{BC}| \Rightarrow L = \sqrt{\frac{\pi^2 EI}{|F_{BC}|}} = 19.1$ in.

Real-world example: Railroad track buckling

What would cause the railroad track to buckle as shown in the picture?

Example 18.4

G. Yang and M.A. Bradford, *Engineering Failure Analysis* 92 (2018), pp. 107-120.