Chapter 5. Stress and strain: generalized concepts

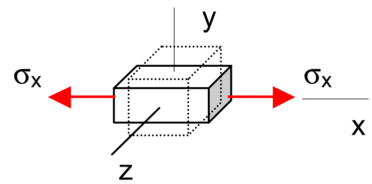
Objectives:
To study the relationships between stress and strain due to a three-dimensional loading of a body.

Background:

- Stress-strain relationship for uni-axial loading (loading in the *x*-direction):

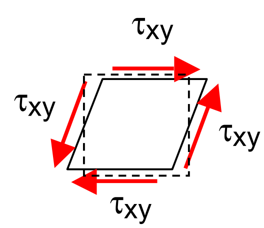
$$\varepsilon_x = \frac{\sigma_x}{E}$$

$$\varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$



- Stress-strain relationship for pure shear loading (loading in *y*-direction on face perpendicular to *x*-axis):

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$



Lecture topics:

a) Resolution of internal forces into normal and tangential (shear) components

b) Thermal strains

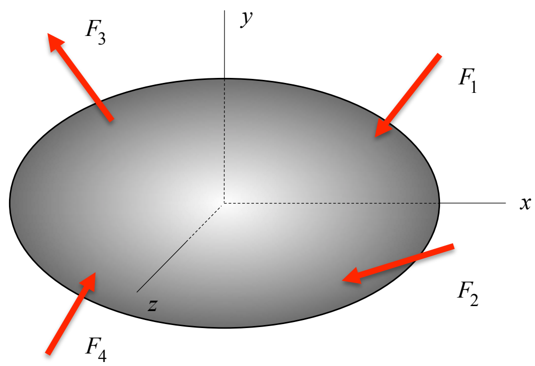
c) Generalized Hooke's law for normal stresses/strains

d) Generalized Hooke's law for shear stresses/strains

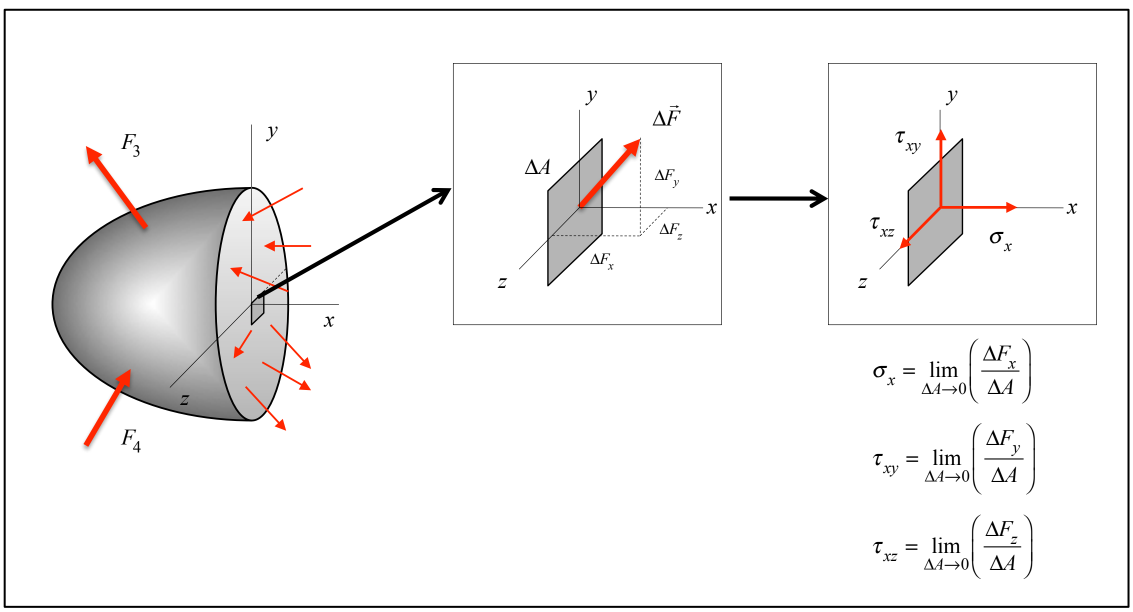
Lecture Notes

a) <u>*Resolution of internal force into normal and tangential (shear) components*</u>

Consider a general 3-D loading on a component:

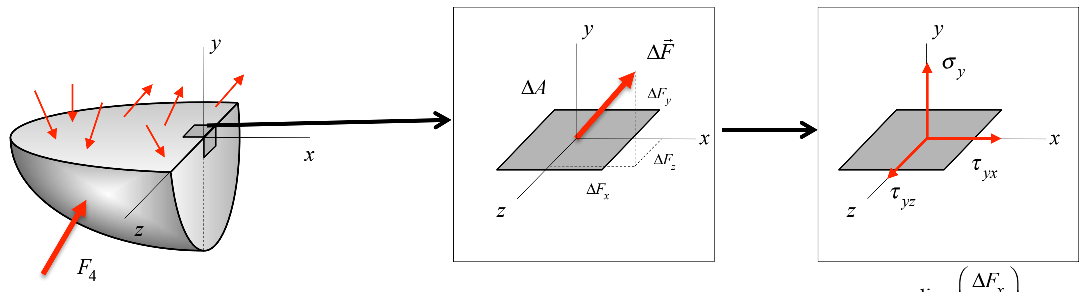


Making cut through body parallel to *yz*-plane:



σ_x is the normal stress on the +*x*-face, and τ_{xy} and τ_{xz} are the components of shear stress on the *x*-face in the *y*- and *z*-directions, respectively.

Next, making a cut through body parallel to *xz*-plane:



$$\tau_{yx} = \lim_{\Delta A \to 0} \left(\frac{\Delta F_x}{\Delta A} \right)$$

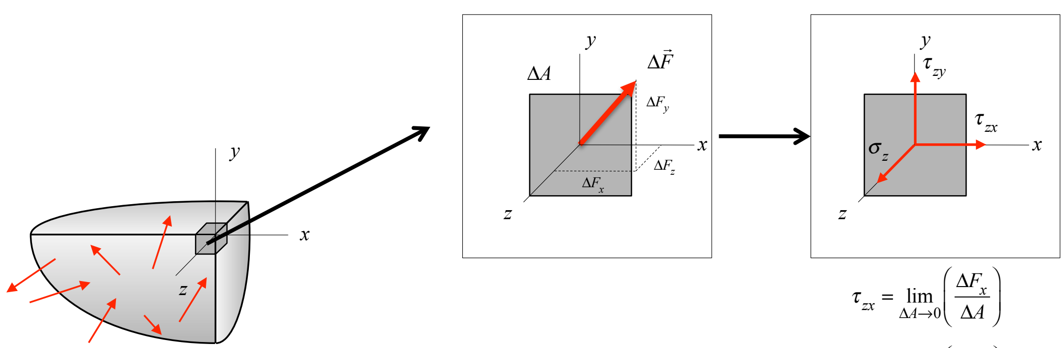
$$\sigma_y = \lim_{\Delta A \to 0} \left(\frac{\Delta F_y}{\Delta A} \right)$$

y

$$\tau_{yz} = \lim_{\Delta A \to 0} \left(\frac{\Delta F_z}{\Delta A} \right)$$

σ_y is the normal stress on the +*y*-face, and τ_{yx} and τ_{yz} are the components of shear stress on the +*y*-face in the *x*- and *z*-directions, respectively.

Next, making a cut through body parallel to *xy*-plane:



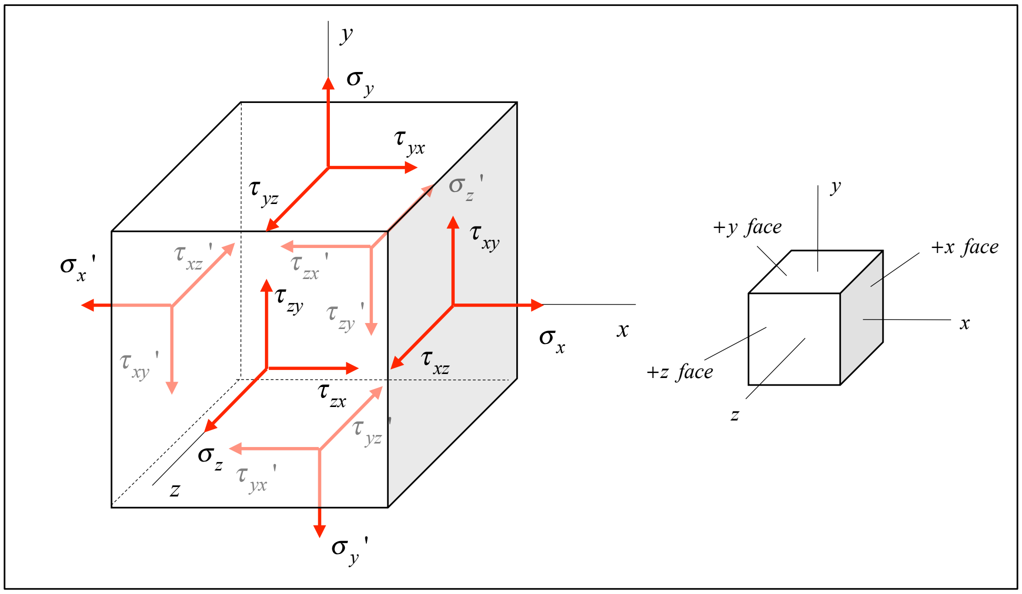
$$\tau_{zx} = \lim_{\Delta A \to 0} \left(\frac{\Delta F_x}{\Delta A} \right)$$

$$\tau_{zy} = \lim_{\Delta A \to 0} \left(\frac{\Delta F_y}{\Delta A} \right)$$

$$\sigma_z = \lim_{\Delta A \to 0} \left(\frac{\Delta F_z}{\Delta A} \right)$$

σ_z is the normal stress on the +*z*-face, and τ_{zx} and τ_{zy} are the components of shear stress on the +*z*-face in the *x*- and *y*-directions, respectively.

Suppose that we continue with an additional set of three cuts through the body at this point of interest, chosen here as the origin of the *xyz*-axes, with these cuts representing the $-x$, $-y$ and $-z$ planes. Following these cuts, we are left of a six-sided "stress element" whose sides are made up of the $\pm x$, $\pm y$ and $\pm z$ faces. As shown below, we have three components of stress (one normal and two shear) on each face.



"stress element" at a point in the body

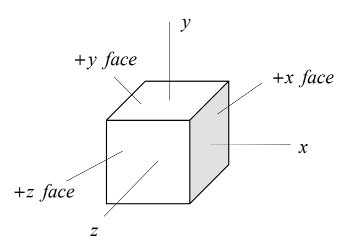
From this, we introduce the concept of a stress element as a cube of infinitesimal dimensions centered on the point of interest in the body.

Naming convention

- σ_i is the normal stress on face "*i*".
- τ_{ij} is the shear stress on face "*i*" in the "j" direction.

Sign convention – signs for components of stress on a stress element

- A normal stress σ_i is positive (negative) if it points outward (inward) on face "*i*" of the element, for $i = x, y, z$. Note that a positive (negative) normal stress corresponds to tension (compression).
- A shear stress τ_{ij} is positive if it points in the positive (negative) *j*-direction on the positive (negative) *i*-face of the stress cube. Otherwise, the shear stress is negative.



Number of unique stress components for a 3D state of stress

In total, we have six components of normal stress:

$$\left(\sigma_x, \sigma_y, \sigma_z, \sigma'_x, \sigma'_y, \sigma'_z\right)$$

and twelve components of shear stress:

$$\left(\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}, \tau'_{xy}, \tau'_{xz}, \tau'_{yx}, \tau'_{yz}, \tau'_{zx}, \tau'_{zy}\right)$$

on the stress element.

Using static equilibrium equations for the cube, the following relations can be derived relating these stresses:

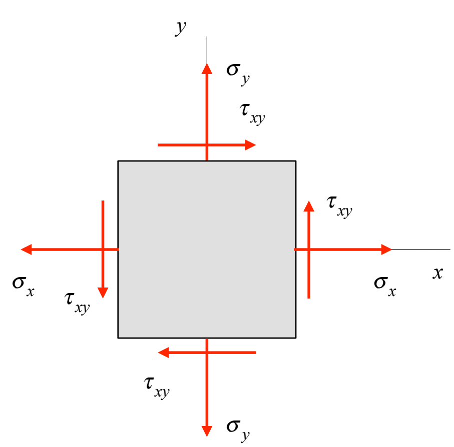
$$\sigma'_x = \sigma_x \qquad \sigma'_y = \sigma_y \qquad \sigma'_z = \sigma_z$$

$$\tau'_{xy} = \tau_{xy} = \tau'_{yx} = \tau_{yx} \qquad \tau'_{yz} = \tau_{yz} = \tau'_{zy} = \tau_{zy} \qquad \tau'_{zx} = \tau_{zx} = \tau'_{xz} = \tau_{xz}$$

In summary, there are, in total, only three unique normal stresses $\left(\sigma_x, \sigma_y, \sigma_z\right)$ and three unique shear stresses acting on the cube $\left(\tau_{xy}, \tau_{xz}, \tau_{yz}\right)$.

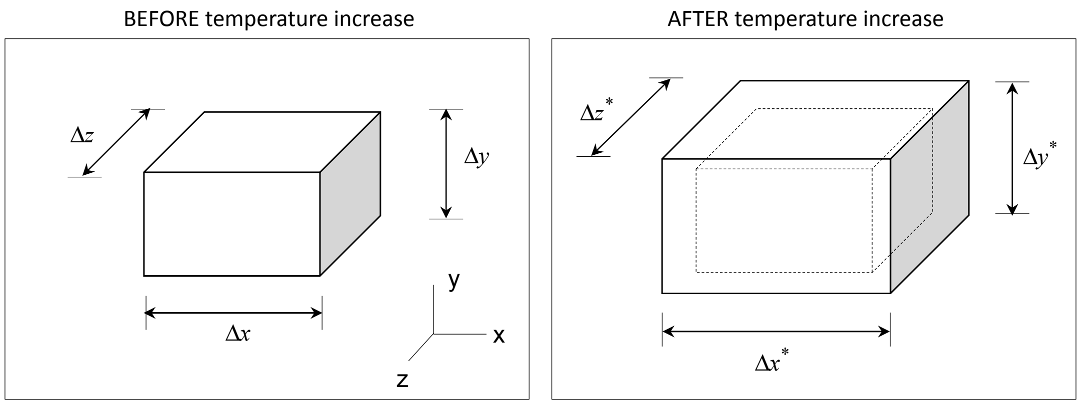
Special case: state of plane stress in the xy-plane

If all components of stress are acting on only the *x*- and *y*-faces, we see the following looking down the *z*-axis. All stress components shown in the figure below are positive according to the sign conventions defined above.



b) *Thermal strains*

As a result of a uniform increase in temperature ΔT, most engineering materials will experience a uniform extensional strain in all three directions. This extensional strain is proportional to the temperature increase ΔT. Consider the cube shown below that is given a uniform temperature increase:



The thermal strains induced by the temperature increase are found from the usual definitions:

$$\varepsilon_{x,T} = \lim_{\Delta x \to 0} \left(\frac{\Delta x^* - \Delta x}{\Delta x} \right)$$

$$\varepsilon_{y,T} = \lim_{\Delta y \to 0} \left(\frac{\Delta y^* - \Delta y}{\Delta y} \right)$$

$$\varepsilon_{z,T} = \lim_{\Delta z \to 0} \left(\frac{\Delta z^* - \Delta z}{\Delta z} \right)$$

Since this thermal strain is uniform and is proportional to ΔT, we can write these as:

$$\varepsilon_{x,T} = \varepsilon_{y,T} = \varepsilon_{z,T} = \alpha \Delta T$$

where α is the coefficient of thermal expansion (having units of 1/°F, or 1/°C).

Note that temperature changes produce only extensional strains (no shear strains).

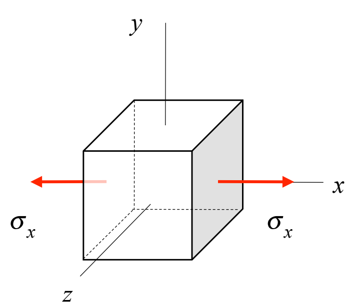
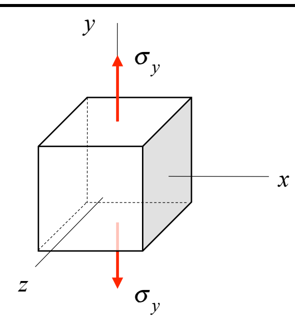
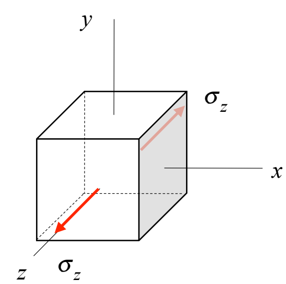
c) *Generalized Hooke's law for normal stresses/strains*

Recall that for uni-axial loading along the x-axis, the normal strains in the x, y and z directions in the body were found to be:

$$\varepsilon_x = \sigma_x / E$$

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x / E$$

where E and ν are the Young's modulus and Poisson's ratio for the material. For a 3-D loading of a body, we have three normal stress components σ_x, σ_y and σ_z acting simultaneously. For this case, we will consider the strains due to each normal component of stress individually and add these together using linear superposition (along with the thermal strains) to determine the resulting three components of strain ε_x, ε_y and ε_z. Consider the individual contributions of the three components of stress shown below:

Strains due to mechanical loading in the x-direction	*Strains due to mechanical loading in the y-direction*	*Strains due to mechanical loading in the z-direction*
		
$\varepsilon_x = \sigma_x / E$	$\varepsilon_x = -\nu\varepsilon_y = -\nu\sigma_y / E$	$\varepsilon_x = -\nu\varepsilon_z = -\nu\sigma_z / E$
$\varepsilon_y = -\nu\varepsilon_x = -\nu\sigma_x / E$	$\varepsilon_y = \sigma_y / E$	$\varepsilon_y = -\nu\varepsilon_z = -\nu\sigma_z / E$
$\varepsilon_z = -\nu\varepsilon_x = -\nu\sigma_x / E$	$\varepsilon_z = -\nu\varepsilon_y = -\nu\sigma_y / E$	$\varepsilon_z = \sigma_z / E$

The total strain in each direction is found through superposition of the individual strains along with the thermal strains. Adding together these components (across each row of the above table) gives:

$$\varepsilon_x = \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y - \frac{\nu}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \alpha\Delta T$$

$$\varepsilon_y = -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y - \frac{\nu}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \alpha\Delta T$$

$$\varepsilon_z = -\frac{\nu}{E}\sigma_x - \frac{\nu}{E}\sigma_y + \frac{1}{E}\sigma_z + \alpha\Delta T = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \alpha\Delta T$$

The above are known as the generalized Hooke's law equations for normal stresses/strains due to 3-D loadings on a body.

Observation:

Note from the preceding equations that thermal strains can exist in the absence of stresses; that is, if $\sigma_x = \sigma_y = \sigma_z = 0$, we still have: $\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha \Delta T$. Later on in the course, we will observe that stresses from thermal loadings can develop only in the presence of mechanical forces. In particular, a body that is heated in the absence of displacement constraints will be stress-free.

(e) Generalized Hooke's law for shear stresses/strains

It can be shown that the three components of shear stress, $\left(\tau_{xy}, \tau_{xz}, \tau_{yz}\right)$ are related to the corresponding shear strains $\left(\gamma_{xy}, \gamma_{xz}, \gamma_{yz}\right)$ by the following equations:

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

$$\gamma_{xz} = \frac{\tau_{xz}}{G}$$

$$\gamma_{yz} = \frac{\tau_{yz}}{G}$$

where G is the shear modulus of the material.

Example 5.3

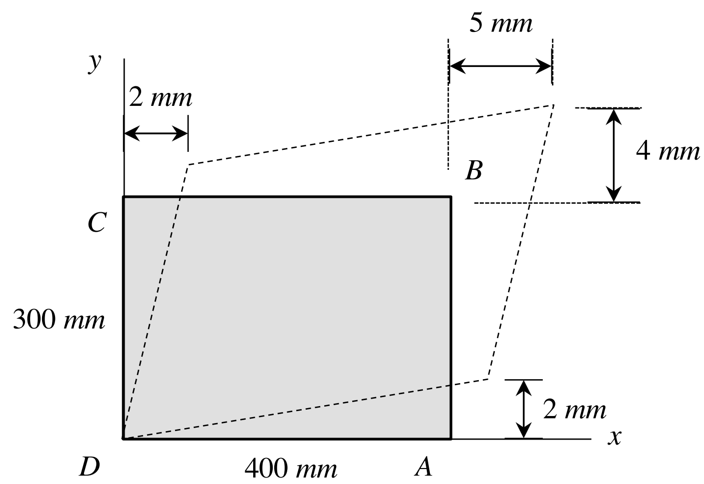
Determine the state of strain that corresponds to the following 3-D state of stress at a certain point in a steel machine component:

$$\sigma_x = 60\ MPa \quad ; \quad \sigma_y = 20\ MPa \quad ; \quad \sigma_z = 30\ MPa$$
$$\tau_{xy} = 20\ MPa \quad ; \quad \tau_{xz} = 15\ MPa \quad ; \quad \tau_{yz} = 10\ MPa$$

Use $E = 210\ GPa$ and $\nu = 0.3$ for steel.

Example 5.4

Recall the general definition of strain. Find the normal and shear strains along x and y at point D.

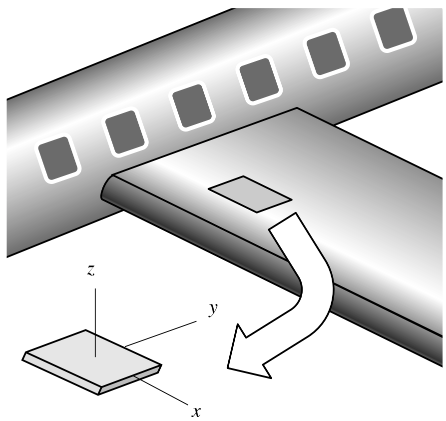


Example 5.6

When thin sheets of material, like the top "skin" of the airplane wing in the following figure is subjected to stresses, they are said to be in a state of plane stress, with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$. For the case that $\Delta T = 0$, show that the Hooke's may can be written as

$$\sigma_x = \frac{E}{1-\upsilon^2}\left(\varepsilon_x + \upsilon\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\upsilon^2}\left(\varepsilon_y + \upsilon\varepsilon_x\right)$$



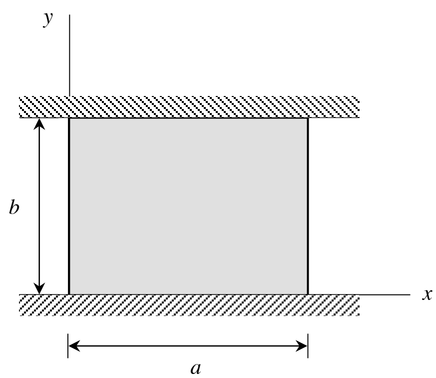
Example 5.8

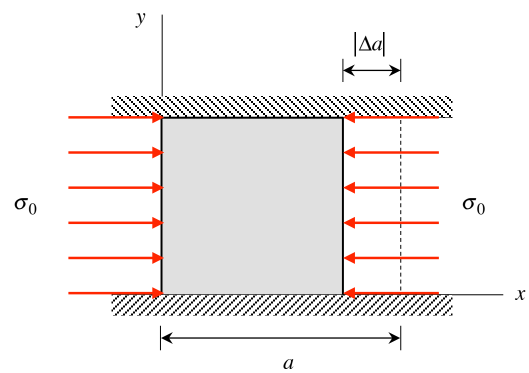
A block of linearly elastic material (E,ν) is compressed between two rigid, perfectly smooth surfaces by an applied stress $\sigma_x = -\sigma_0$. A non-zero stress σ_y is induced by the restraining surfaces at $y = 0$ and $y = b$.

(a) Determine the value of the restraining stress σ_y.

(b) Determine Δa, the change in the x dimension of the block.

(c) Determine the change Δt in the thickness in the z direction.



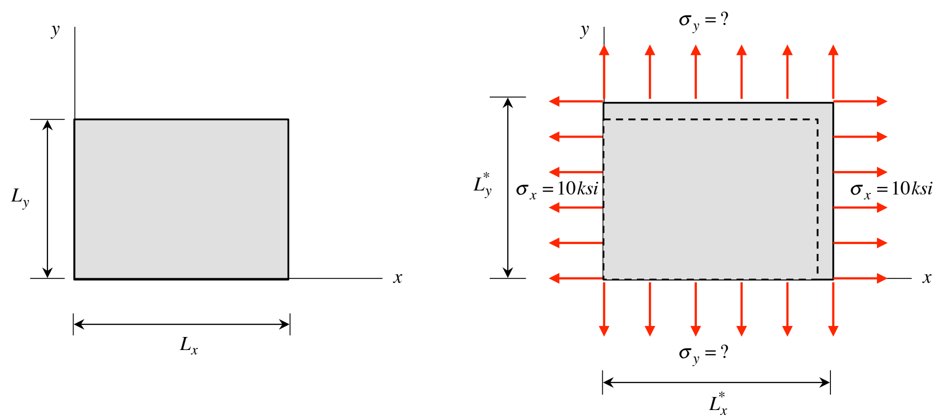


Example 5.9

A thin, rectangular plate is subjected to a uniform biaxial state of stress $\left(\sigma_x, \sigma_y\right)$. All other components of stresses are zero. The initial dimensions of the place at $L_x = 4$ in. and $L_y = 2$ in., but after the loading is applied, the dimensions are L_x^* = 4.00176 in., and L_y^* = 2.00344 in. If it is known that σ_x = 10 ksi and E = 10×10^3 ksi:

(a) What is the value of the Poisson's ratio?

(b) What is the value of σ_y ?



Additional notes: