11. Beams: Deflections

Objectives:
To study the transverse deflections of beams and the application of beam deflection analysis to the stress analysis of indeterminate beams.

Background:

- Fundamental equations relating transverse loading p, shear force V and bending moment M for beams:

$$\frac{dV}{dx} = p(x)$$

$$\frac{dM}{dx} = V(x)$$

- *Moment-curvature* equation for deflection of beams:

$$M = \frac{EI}{\rho}$$

where ρ is the radius of curvature of deflection curve for beam.

Lecture topics:

a) Calculation of beam deflection for *statically-determinate* beams using 2nd-order and 4th-order integration methods.

b) Calculation of beam deflection for *statically-indeterminate* beams while simultaneously solving for the unknown reactions on the beam.

c) Using *superposition methods* for determining beam deflections.

Lecture Notes

Recall from the last section of notes the following three fundamental relationships relating the shear force V, bending moment M and the applied force/length p(x) acting on a thin beam:

$$\frac{dV}{dx} = p(x) \quad (1)$$

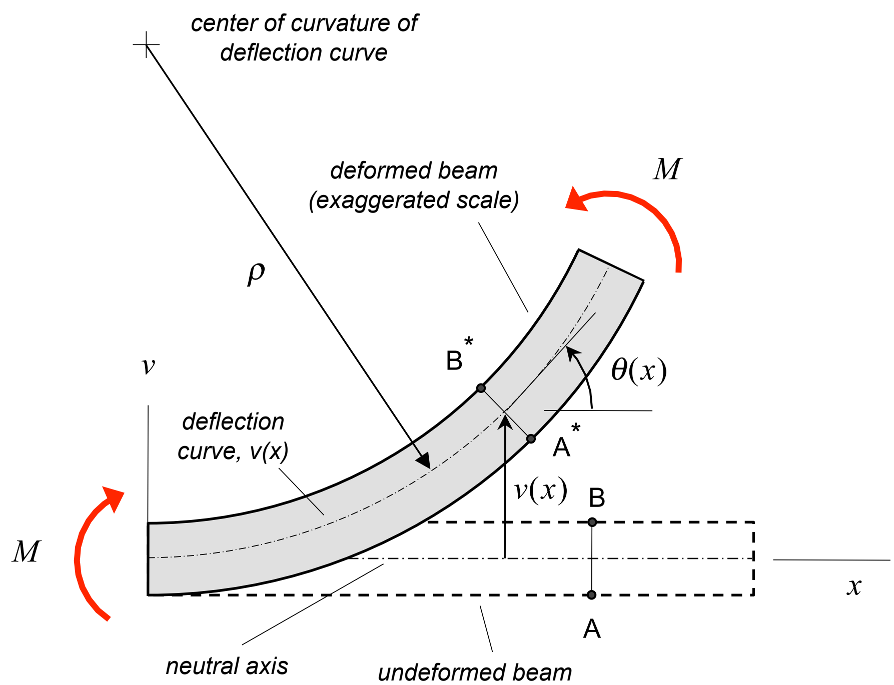
$$\frac{dM}{dx} = V \quad (2)$$

$$M = \frac{EI}{\rho} \quad (3)$$

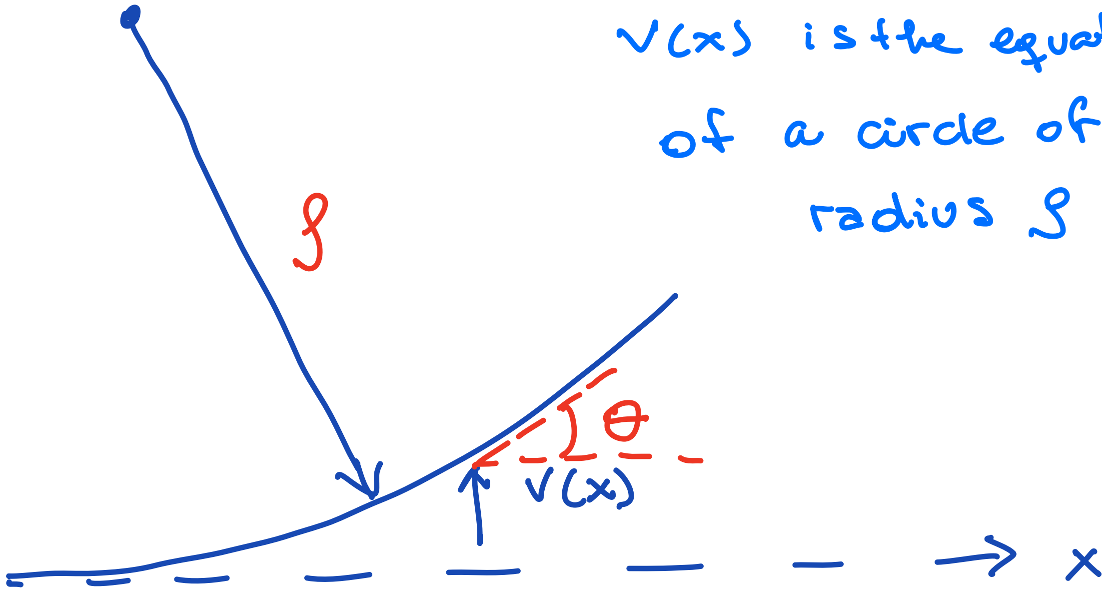
where E is the Young's modulus of the material, I is the second area moment of the beam and ρ is the radius of curvature of the deflection curve $v(x)$ of the neutral axis of the beam. Combining equations (1)-(3) gives:

$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = p(x) \quad \Rightarrow \quad \frac{d^2}{dx^2}\left(\frac{EI}{\rho}\right) = p(x) \quad (4)$$

Consider the figure below showing the deformation of a beam in bending where $v(x)$ is the transverse deflection of the neutral axis of the beam.



In this figure, $\theta(x)$ is the angle of rotation of the cross section of the beam and $\rho(x)$ is the radius of curvature of the deflection curve $v(x)$. From basic calculus, $\theta(x)$ and $\rho(x)$ can be expressed in terms of the deflection curve $v(x)$ as:



v(x) is the equation of a circle of radius ρ

$$(1) \quad v(x) = \sqrt{\rho^2 - x^2}$$

small angle

$$\frac{dv}{dx} = \frac{\Delta v}{\Delta x} = \tan\theta \simeq \theta$$

$$\frac{dv}{dx} = \theta \rightarrow$$ slope (angle of rotation)

From (1)

$$\frac{dv(x)}{dx} = \frac{-x}{\sqrt{\rho^2 - x^2}} \qquad \frac{d^2v}{dx^2} = \frac{-\rho^2}{(\rho^2 - x^2)^{3/2}}$$

Solve for ρ

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{(1 + (dv/dx)^2)^{3/2}}$$

$$tan\theta = \frac{dv}{dx}$$

$$\frac{1}{\rho} = \frac{d^2v / dx^2}{\left[1+\left(dv/dx\right)^2\right]^{3/2}}$$

$\frac{dv}{dx} << 1 \quad \left(\frac{dv}{dx}\right)^2 \approx 0$

If we restrict our considerations deflections with small slopes ($dv/dx << 1$), the above reduce to:

$$\frac{dv}{dx} = \theta \tag{5}$$

$$\frac{1}{\rho} = \frac{d^2v}{dx^2} \tag{6}$$

Substitution of equation (6) into equations (2), (3) and (4) gives the following:

$$M(x) \quad M = EI\frac{d\theta}{dx} = EI\frac{d^2v}{dx^2} \quad ; \quad moment - curvature\ equation \tag{7}$$

$$V(x) \quad V = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) \quad ; \quad shear - deflection\ equation \tag{8}$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = p(x) \quad ; \quad load - deflection\ equation \tag{9}$$

(*)

In addition, we also have equation (5) for the angle of rotation θ in terms of the beam deflection.

Note that if the properties of the beam are constant along its length (i.e., $EI = constant$), then equations (8) and (9) reduce to:

$$V = EI\frac{d^3v}{dx^3} \tag{8a}$$

$$EI\frac{d^4v}{dx^4} = p(x) \tag{9a}$$

(*) these are O.D.E that we can solve to obtain $v(x)$ if we know $V(x)$ or $p(x)$

(7) 2nd order ODE
(8) 3rd order ODE
(9) 4th order ODE

→ need B.C on $v(x)$ or $\frac{dv}{dx} = \theta(x)$

Equation (9a) can be solved for the deflection $v(x)$ through four integrations with the enforcement of boundary conditions in the evaluation of the resulting constants of integration. A list of common boundary conditions are presented in the following table.

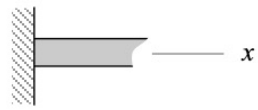
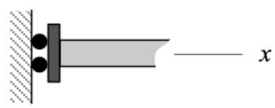
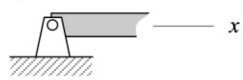
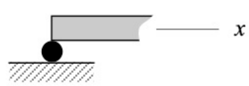
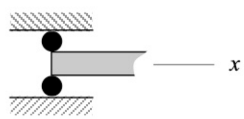
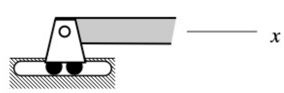
fixed support		$v = 0$ $\theta = \frac{dv}{dx} = 0$
constrained rotation support		$\theta = \frac{dv}{dx} = 0$ $V = 0$ → $\frac{d^3v}{dx^3} = 0$
free end		$V = 0$ $M = 0$ → $\frac{d^2v}{dx^2} = 0$
simple support	*pin joint* 	$v = 0$ $M = 0$
	roller 	
	double roller 	
	roller-in-slot 	

TABLE 7.1 Boundary Conditions

Type	Symbol*	2nd Order	4th Order
Fixed end		$v = 0$ $v' = 0$	$v = 0$ $v' = 0$
Simple support		$v = 0$	$v = 0$ $M = 0$
Free end		No BC	$V = 0$ $M = 0$
Concentrated force	P_0	No BC	$V = P_0$ $M = 0$
Concentrated couple	M_0	No BC	$V = 0$ $M = -M_0$

*These boundary conditions also apply if the boundary under consideration is the other end of the beam (i.e., $x = L$).

TABLE 7.2 Continuity Conditions*

Type	Symbol	2nd Order	4th Order
Roller	① ②	$v_1 = v_2 = 0$ $v_1' = v_2'$	$v_1 = v_2 = 0$ $v_1' = v_2'$ $M_1 = M_2$
Discontinuity in load function		$v_1 = v_2$ $v_1' = v_2'$	$v_1 = v_2$ $v_1' = v_2'$ $V_1 = V_2$ $M_1 = M_2$
Concentrated force	P_0	$v_1 = v_2$ $v_1' = v_2'$	$v_1 = v_2,\ \ v_1' = v_2'$ $V_2 - V_1 = P_0$ $M_1 = M_2$
Concentrated couple	M_0	$v_1 = v_2$ $v_1' = v_2'$	$v_1 = v_2,\ \ v_1' = v_2'$ $V_1 = V_2$ $M_2 - M_1 = -M_0$
Pin, with force	P_0	$v_1 = v_2$	$v_1 = v_2$ $V_2 - V_1 = P_0$ $M_1 = M_2 = 0$

*The displacement (v) and slope (v') continuity conditions that are listed in Table 7.2 are obtained by inspection, that is, by simply looking at the figures in the "Symbol" column. The continuity conditions on shear force (V) and bending moment (M) are obtained by taking a local free-body diagram of the "joint" that is common to beam segments (1) and (2).

Deflections of statically-determinate beams – DEFINITE INTEGRAL APPROACH

Recall that for statically-determinate beams, we can determine the external reactions on the beam using the rigid body equilibrium equations. Assume that for a given determinate problem we have already determined these external reactions through equilibrium analysis. Using these, our goal is to determine the deflection of the beam over the full length of the beam.

To this end, we will now reconsider equations (1), (2), (5) and (7) above. We will integrate these equations over a given segment $x_1 < x < x_2$ of the beam. Note that the following results assume that the cross sectional and material properties are constant throughout a given segment.

<u>Equation (1)</u>:

$$\frac{dV}{dx} = p(x) \qquad \Rightarrow \qquad V(x) = V(x_1) + \int_{x_1}^{x} p(s)\, ds \tag{10}$$

<u>Equation (2)</u>:

$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M(x) = M(x_1) + \int_{x_1}^{x} V(s)\, ds \tag{11}$$

<u>Equation (7)</u>:

$$EI\frac{d\theta}{dx} = M(x) \qquad \Rightarrow \qquad \theta(x) = \theta(x_1) + \frac{1}{EI}\int_{x_1}^{x} M(s)\, ds \tag{12}$$

<u>Equation (5)</u>:

$$\frac{dv}{dx} = \theta(x) \qquad \Rightarrow \qquad v(x) = v(x_1) + \int_{x_1}^{x} \theta(s)\, ds \tag{13}$$

These results can be used two alternate ways for determining the deflection of a beam:

i) *<u>Fourth-order approach</u>* – Here we start with the loading $p(x)$ and perform the four integrations of (10)-(15) to obtain v(x).

ii) *<u>Second-order approach</u>* – Here we determine bending moment distribution $M(x)$ through FBDs and equilibrium analysis. With this $M(x)$, equations (12)-(13) are used to produce the deflection $v(x)$.

Note that with this *definite integral* approach, the boundary conditions such as $\theta(x_1)$ and $v(x_1)$ naturally appear in the solutions.

Deflections of statically-determinate beams – INDEFINITE INTEGRAL APPROACH

Equations (10)-(13) can be combined into a single differential equation as follows. Taking a derivative of (13) with respect to *x*:

$$\frac{d\theta}{dx} = \frac{d^2 v}{dx^2} \qquad (13a)$$

and substituting this into (12) gives:

$$\boxed{M(x) = EI\frac{d^2 v}{dx^2}} \qquad (12a)$$

Next, taking a derivative of (12a) with respect to x:

$$\frac{dM}{dx} = \frac{d}{dx}\left(EI\frac{d^2 v}{dx^2} \right)$$

and substituting this into (11) gives:

$$V = \frac{d}{dx}\left(EI\frac{d^2 v}{dx^2} \right) \qquad (11a)$$

And, finally taking a derivative of (11a) with respect to x:

$$\frac{dV}{dx} = \frac{d^2}{dx^2}\left(EI\frac{d^2 v}{dx^2} \right)$$

and substituting this into (10) gives:

$$\boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2 v}{dx^2} \right) = p(x)} \qquad (10a)$$

These results can be used two alternate ways for determining the deflection of a beam:

i) <u>*Fourth-order approach*</u> - For a given loading p(x), equation (10a) is integrated *four* times to produce the deflection v(x).

ii) <u>*Second-order approach*</u> – Here the bending moment distribution $M(x)$ is determined through FBDs and equilibrium analysis. With this $M(x)$, equation (12a) is integrated *two* times to produce the deflection $v(x)$.

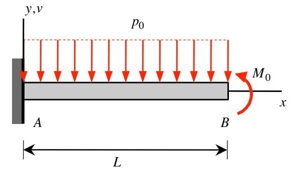
With this approach, constants of integration are introduced at each step. These constants are evaluated by enforcing boundary conditions. For example, consider the second order approach:

$$\frac{dv}{dx} = c_1 + \int \frac{M(x)}{EI} dx$$

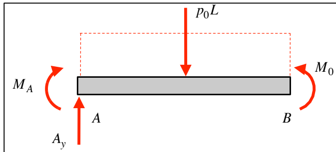
$$v(x) = c_2 + c_1 x + \int \left(\int \frac{M(x)}{EI} dx \right) dx$$

Here with the *indefinite integral* approach, we need to recall which boundary conditions are needed to be enforced to find the integration constants c_1 and c_2. From before, we saw that with the definite integral approach, the necessary boundary conditions naturally appeared in the process. *The advantage goes to the definite integral approach!*

Example 11.1 Determine the deflection curve v(x) and the beam rotation angle at end B.



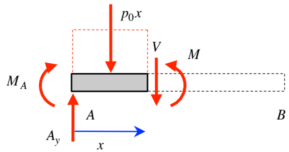
determine reactions



$$\sum F_y = A_y - p_0 x - V = 0 \quad \Rightarrow \quad A_y = p_0 L$$

$$\sum M_A = -M_A - (p_0 L)(L/2) + M_0 = 0 \quad \Rightarrow \quad M_A = M_0 - p_0 L^2/2$$

2nd order method



$$\sum F_y = A_y - p_0 x - V = 0 \quad \Rightarrow \quad V(x) = A_y - p_0 x$$

$$\sum M_A = -M_A - (p_0 x)\left(\frac{x}{2}\right) - Vx + M = 0 \quad \Rightarrow \quad M(x) = M_A + A_y x - \frac{1}{2} p_0 x^2$$

4th order method

$$V(x) = V(0) + \int_0^x p(x)\,dx = A_y - p_0 x$$

$$M(x) = M(0) + \int_0^x V(x)\,dx = M_A + A_y x - \frac{1}{2} p_0 x^2$$

slopes and deflections

$$\theta(x) = \theta(0) + \frac{1}{EI}\int_0^x M(x)\,dx = \frac{1}{EI}\left[M_A x + \frac{1}{2}A_y x^2 - \frac{1}{6}p_0 x^3\right]$$

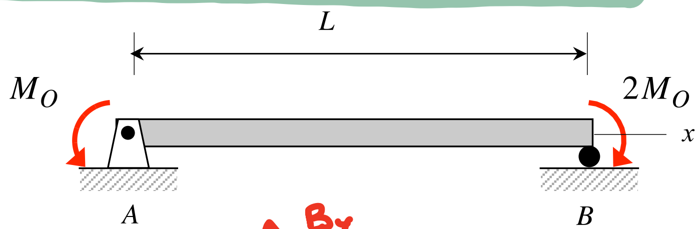
$$v(x) = v(0) + \int_0^x \theta(x)\,dx = \frac{1}{EI}\left[\frac{1}{2}M_A x^2 + \frac{1}{6}A_y x^3 - \frac{1}{24}p_0 x^4\right] = \frac{1}{EI}\left[\frac{1}{2}(M_0 - p_0 L^2/2)x^2 + \frac{1}{6}(p_0 L)x^3 - \frac{1}{24}p_0 x^4\right]$$

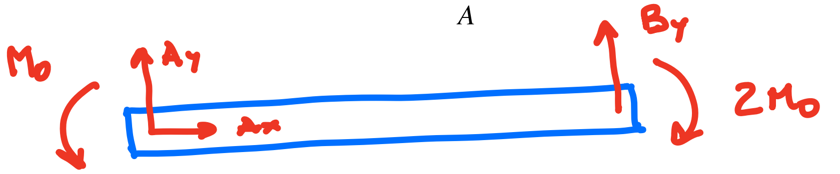
$$\theta(L) = \frac{1}{EI}\left[M_A L + \frac{1}{2}A_y L^2 - \frac{1}{6}p_0 L^3\right] = \frac{1}{EI}\left[\left(M_0 - p_0 L^2/2\right)L + \frac{1}{2}(p_0 L)L^2 - \frac{1}{6}p_0 L^3\right] = \frac{1}{EI}\left[M_0 L - \frac{1}{6}p_0 L^3\right]$$

Example 11.2

The beam is made up of a material with an elastic modulus E and has a cross-sectional second area moment I, both of which are constant along the length of the beam. Determine the beam rotation at end A and the deflection at x = L/2.

B.C $v(0) = 0$, $v(L) = 0$





$A_x = 0$

$\Sigma M_A = 0$

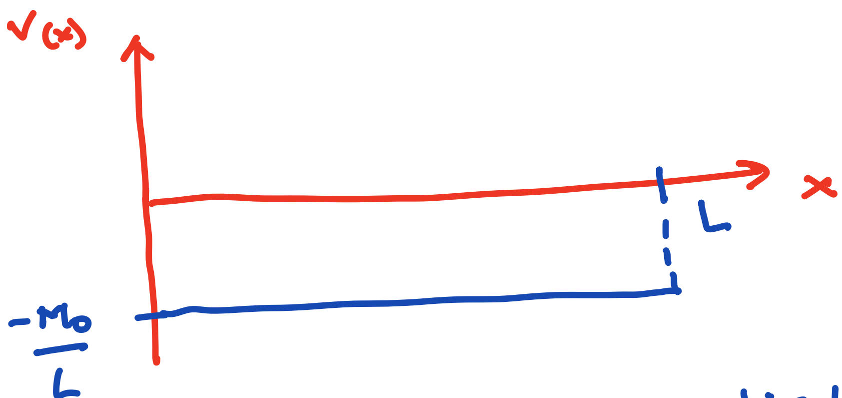
$M_o + B_y L - 2M_o = 0$

$B_y = \frac{M_o}{L}$

$\Sigma F_y = 0$

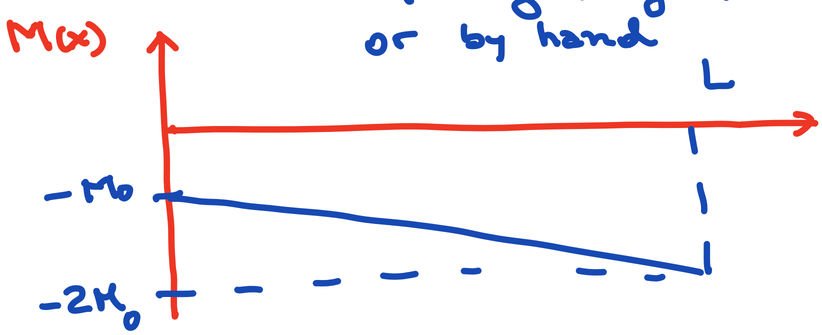
$A_y + B_y = 0$

$A_y = -\frac{M_o}{L}$



$$V(x) = -\frac{M_o}{L}$$

integrate graphically or by hand



$$M(x) = -M_o - \frac{M_o}{L}x$$

Approach 1 You have $M(x) = -M_o - \frac{M_o}{L}x$

Use equation (7) and definite integrals.

(7) $EI \dfrac{d\theta}{dx} = EI \dfrac{d^2v}{dx^2} = M(x)$

$$EI \frac{d\theta}{dx} = M(x)$$

$$EI \int_0^x \frac{d\theta}{d\xi} d\xi = \int_0^x M(\xi)\, d\xi$$

$$EI\left(\theta(x) - \theta(0)\right) = \int_0^x \left(-M_0 - \frac{M_0}{L}\xi\right) d\xi$$

$$EI\,\theta(x) = EI\,\theta(0) + \left(-M_0 \xi - \frac{M_0}{2L}\xi^2\right)\Big|_0^x$$

$$\theta(x) = \underbrace{\theta(0)}_{\theta_A} + \frac{1}{EI}\left(-M_0 x - \frac{M x^2}{2L}\right)$$

$$\frac{dv}{dx} = \theta_A + \frac{1}{EI}\left(-M_0 x - \frac{M_0 x^2}{2L}\right)$$

$$v(x) - v(0) = \int_0^x \theta_A + \frac{1}{EI}\left(-M_0 \xi - \frac{M_0 \xi^2}{2L}\right) d\xi$$

$$v(x) = \cancel{v(0)}^{=0} + \theta_A x + \frac{1}{EI}\left(-\frac{M_0 x^2}{2} - \frac{M_0 x^3}{6L}\right)$$

From B.C $v(0) = 0$

$$v(L) = 0$$

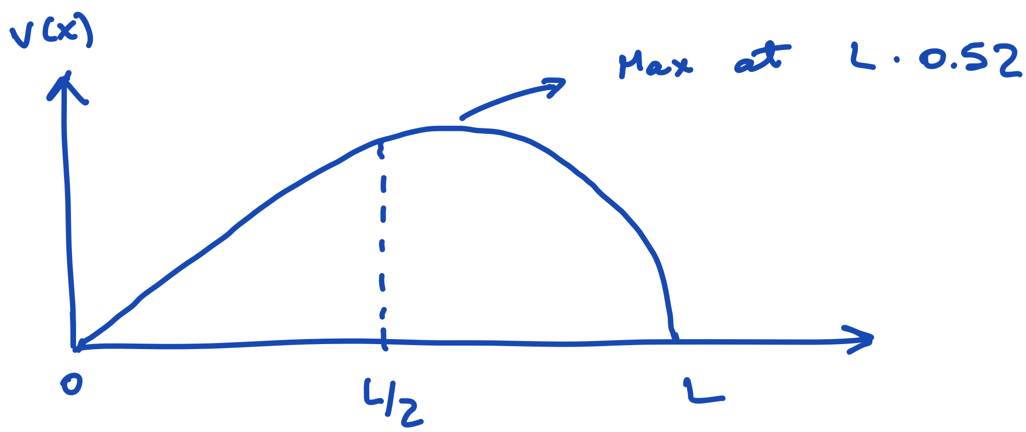
$$v(L) = \theta_A L + \frac{1}{EI}\left(-\frac{M_0 L^2}{2} - \frac{M_0 L^3}{6L}\right) = 0$$

Solve $\theta_A = \frac{2}{3}\frac{M_0 L}{EI}$

$$v(x) = \frac{M_0}{EI} L^2 \left[\left(\frac{x}{L}\right)\frac{2}{3} - \left(\frac{x}{L}\right)^2 \frac{1}{2} - \left(\frac{x}{L}\right)^3 \frac{1}{6}\right]$$

$$v\left(\frac{L}{2}\right) = \frac{M_0 L^2}{EI}\left(\frac{1}{3} - \frac{1}{8} - \frac{1}{48}\right)$$

$$v\left(\frac{L}{2}\right) = \frac{3}{16}\frac{M_0 L^2}{EI}$$



Approach 2   Start with $M(x)$ and use indefinite integrals

(7) $$EI \frac{d\theta}{dx} = EI \frac{d^2 v}{dx^2} = M(x)$$

$$EI \int \frac{d\theta}{dx} dx = \int \left(M_0 - \frac{M_0}{L} x \right) dx$$

$$\theta(x) = \frac{M_0}{EI}\left(x - \frac{x^2}{2L} + A \right)$$

$$\frac{dv}{dx} = \frac{M_0}{EI}\left(x - \frac{x^2}{2L} + A \right)$$

$$v(x) = \frac{M_0}{EI}\left(\frac{x^2}{2} - \frac{x^3}{6L} + Ax + B \right)$$

$$v(x) = +\frac{M_0}{EI} L^2 \left(\left(\frac{x}{L}\right)^2 \frac{1}{2} - \left(\frac{x}{L}\right)^3 \frac{1}{6} + A x + B \right)$$

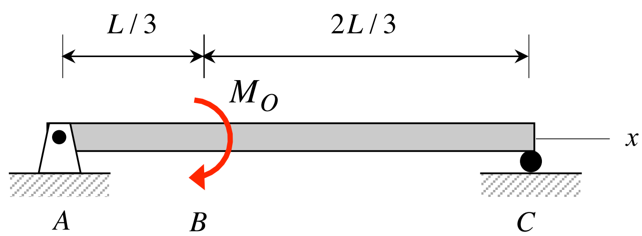
$v(0) = 0 \Rightarrow B = 0$

$v(L) = 0 \Rightarrow A = \frac{2}{3L}$

We recover the same solution

Example 11.3

The simply-supported beam is loaded with a concentrated couple M_0 at B. The beam is made up of a material with an elastic modulus E and has a cross-sectional second area moment I, both of which are constant along the length of the beam. Determine the deflection curve for the beam shown below.



A.1 Geometric properties of plane areas

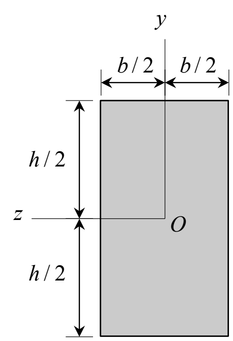
In the following, O, A, I_y and I_z represent the centroid, the area, the second area moment about the y-axis, and the second area moment about the z-axis, respectively, of the plane area shown.

Rectangle

$$A = bh$$

$$I_y = \frac{1}{12} b^3 h$$

$$I_z = \frac{1}{12} b h^3$$

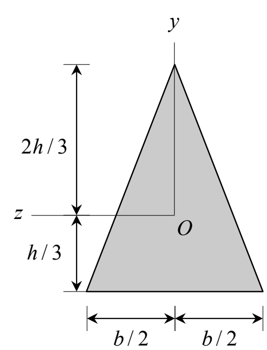


Triangle

$$A = \frac{1}{2} bh$$

$$I_y = \frac{1}{48} b^3 h$$

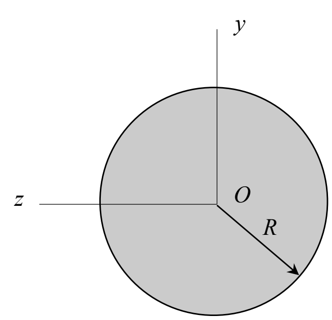
$$I_z = \frac{1}{36} b h^3$$



Circle

$$A = \pi R^2$$

$$I_y = I_z = \frac{\pi}{4} R^4$$

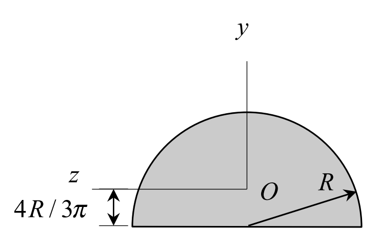


Semi-circle

$$A = \frac{\pi}{2} R^2$$

$$I_y = \frac{\pi}{8} R^4$$

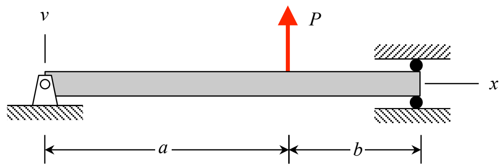
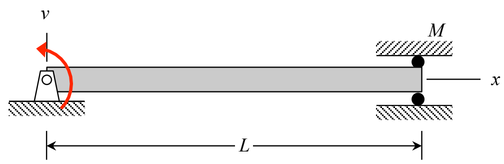
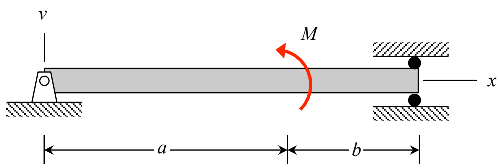
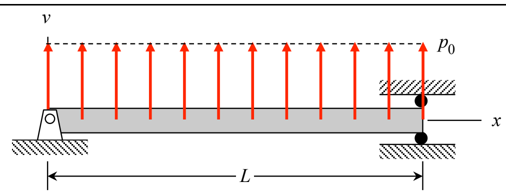
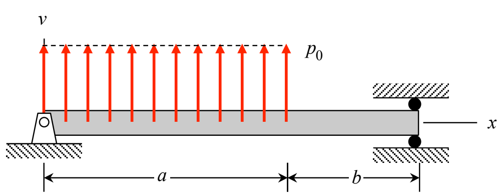
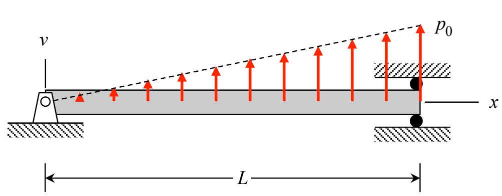
$$I_z = \left(\frac{9\pi^2 - 64}{72\pi} \right) R^4$$



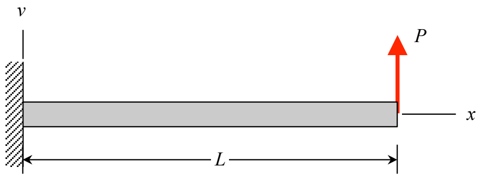
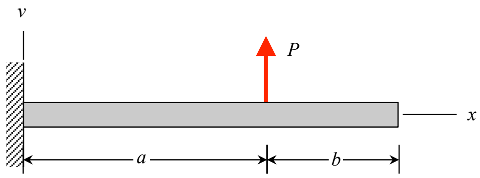
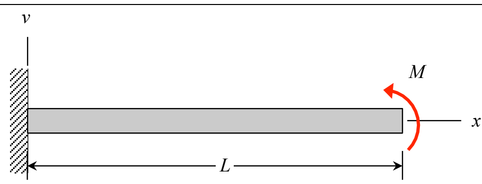
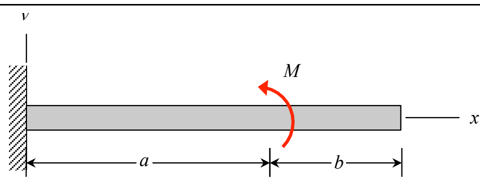
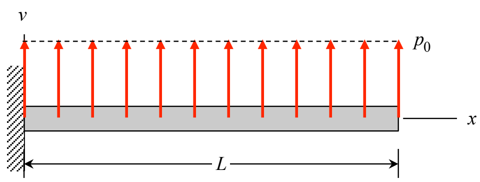
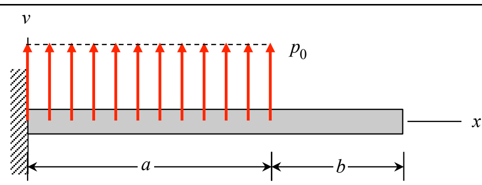
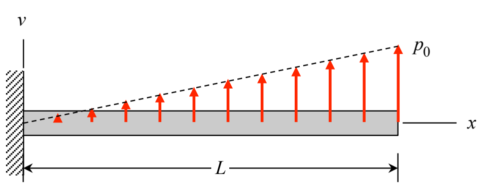
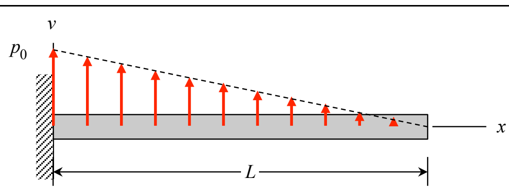
A.2 Beam deflection equations

Formulas are provided below for selected beams and beam loadings, where *EI* is the flexural rigidity for the beam material/cross section and *L* is the beam length.

SIMPLY-SUPPORTED BEAMS

Loading on beam	Deflection equation
	$v(x)=\frac{1}{6}\left[bx\left(L^2-b^2-x^2\right)\right]\frac{P}{LEI}\quad ; \quad 0<x<a$
	$v(x)=\frac{1}{6}\left[x\left(2L^2-3Lx+x^2\right)\right]\frac{M}{LEI}$
	$v(x)=\frac{1}{6}\left[x\left(-6aL+3a^2+2L^2-x^2\right)\right]\frac{M}{LEI}\quad ; \quad 0<x<a$
	$v(x)=\frac{1}{24}\left[x\left(L^3-2Lx^2+x^3\right)\right]\frac{p_0}{EI}$
	$v(x)=\frac{1}{24}\left[x\left(a^4-4a^3L+4a^2L^2+2a^2x^2-4aLx^2+Lx^3\right)\right]\frac{p_0}{LEI}\quad ; \quad 0<x<a$ $=\frac{1}{24}\left[a^2\left(-a^2L+4L^2x+a^2x-6Lx^2+2x^3\right)\right]\frac{p_0}{LEI}\quad ; \quad a<x<L$
	$v(x)=\frac{1}{360}\left[x\left(7L^4-10L^2x^2+3x^4\right)\right]\frac{p_0}{LEI}$

CANTILEVERED BEAMS

Loading on beam	***Deflection equation***
	$v(x)=\frac{1}{6}\left[x^2(3L-x)\right]\frac{P}{EI}$
	$v(x)=\frac{1}{6}\left[x^2(3a-x)\right]\frac{P}{EI}\quad ;\ 0<x<a$ $=\frac{1}{6}\left[a^2(3x-a)\right]\frac{P}{EI}\quad ;\ a<x<L$
	$v(x)=\frac{1}{2}\left[x^2\right]\frac{M}{EI}$
	$v(x)=\frac{1}{2}\left[x^2\right]\frac{M}{EI}\quad ;\ 0<x<a$ $=\frac{1}{2}\left[a(2x-a)\right]\frac{M}{EI}\quad ;\ a<x<L$
	$v(x)=\frac{1}{24}\left[x^2\left(6L^2-4Lx+x^2\right)\right]\frac{p_0}{EI}$
	$v(x)=\frac{x^2}{24}\left[6a^2-4ax+x^2\right]\frac{p_0}{EI}\quad ;\ 0<x<a$ $=\frac{a^3}{24}\left[4x-a\right]\frac{p_0}{EI}\quad ;\ a<x<L$
	$v(x)=\frac{1}{120}\left[x^3\left(20L^3-10L^2x+x^3\right)\right]\frac{p_0}{LEI}$
	$v(x)=\frac{1}{120}\left[x^2\left(10L^3-10L^2x+5Lx^2-x^3\right)\right]\frac{p_0}{LEI}$