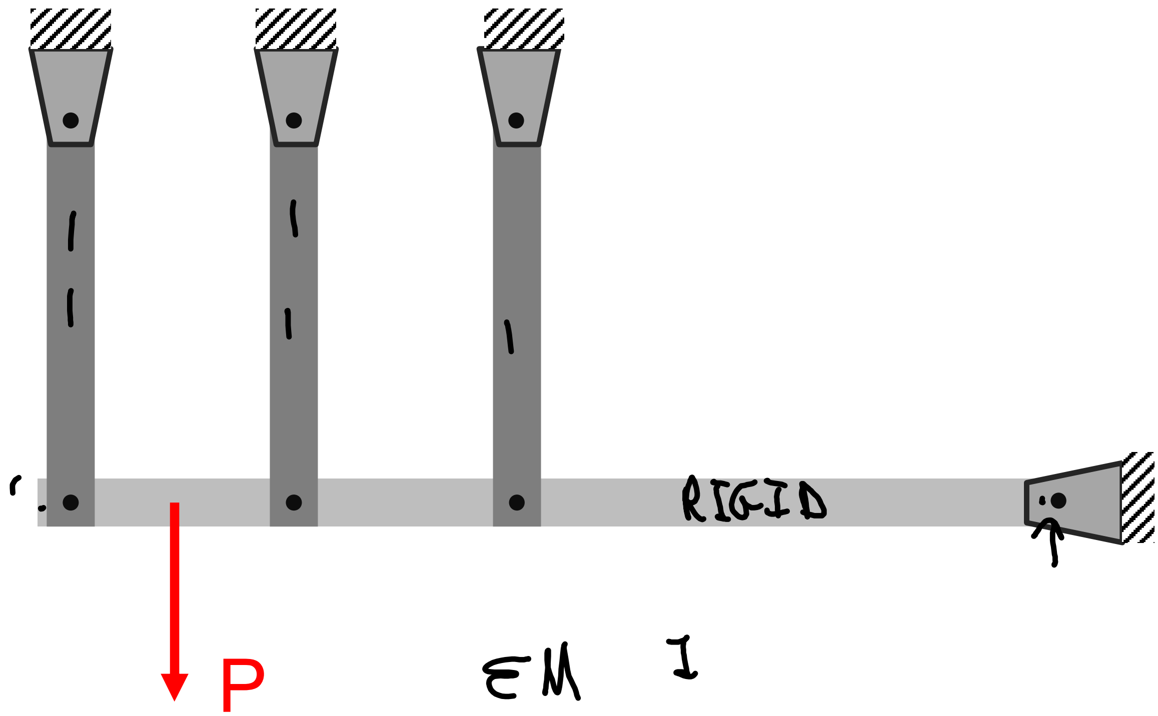
Lecture 7 review question

What is the degree of indeterminacy of this structure?



ΣM 1

~~ΣF_y~~

3 unknowns

1 equ

2

Summary: Indeterminate axial problems (the four-step method)

1. *Draw FBD's and write down equilibrium equations.*
   a. Break system into "elements". For problems in this course, an element:
      i. has forces acting only at its ends,
      ii. has a constant cross section, and
      iii. has constant material properties.
   b. It is recommended that you always draw end forces on element corresponding to "tension" (if the force actually corresponds to compression, you will get a negative value for the force in the end...trust the math...it works!).
   c. Be sure to abide by Newton's 3rd Law (reactions appear in equal and opposite pairs) when drawing your FBD's

2. *Write down the elemental force-deformation equations.*
   a. For the jth element:
   $$e_j = \frac{P_j L_j}{E_j A_j}$$
   b. If you draw all element forces as in tension (as recommended in 1b) above), then P_j has a positive sign in the equation above. If you choose to draw the elemental force as in compression, then P_j has a negative sign.

3. *Write down appropriate compatibility equations*
   a. In this step you will write down the constraint equations that exists among the element elongations.
   b. This step is *problem-dependent* (and requires the most thought):
      i. For an axial system constrained between rigid supports, the compatibility equation needed is that the sum of all the elemental elongations is zero.
      ii. For elements attached to a rigid member, the motion of the rigid member dictates the relationship that exists among the elemental elongations.
      iii. For truss elements, a trigometric relationship must be used to relate the elemental elongations.

Draw deformed geometry.

4. *Solve equations derived in Steps 1-3 for the elemental forces* P_j. Count your number of equations and number of unknowns. If you have sufficient equations to solve, then solve. If not, review the first three steps to see if you are missing needed equilibrium, force/deformation or compatibility equations. From these forces determine at this step, the elemental stresses, strains and elongations can be computed as needed.

Remarks

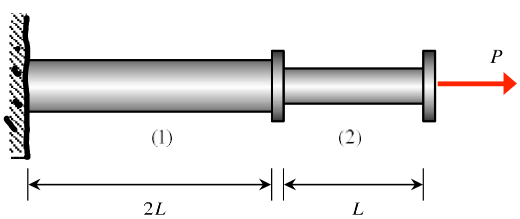
a) If you have enough equations from Step 1 for the number of unknowns, then the problem is *determinate*. That is, you can solve for elemental forces independent of displacement analysis (force equations and elongation equations can be solved *sequentially*).

b) For indeterminate problems, Steps 1-3 produce a set of *coupled* equations. That is, you must *simultaneously* solve for elemental forces and elemental elongations.

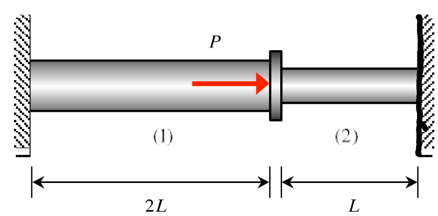
c) Physically, an indeterminate problem is one for which the deformations created by the loading influence the reaction forces. A determinate problem is one for which the reaction forces can be found by treating the elements as being rigid.

To help see the point made in c) above in contrasting determinate and indeterminate problems, consider Rods I and II below. Rod I is determinate. We can determine the loads carried by segments (1) and (2) directly from equilibrium analysis. No information is needed on the material makeup of the segments in calculating either the loads or the stresses in the segments. That is, the stress in the segments does not depend on whether they are made up of steel, aluminum, plastic, or whatever.

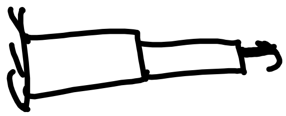
Rod II, however, is indeterminate. We cannot determine the segment loads (and stress) directly from equilibrium. We need to perform a deformation analysis using appropriate compatibility conditions in order to determine these loads. The answers will depend on the deformations, and, therefore will depend of the material makeup. That is, the loads in the segments are different for a steel material for both segments than for one segment made from steel and the other segment made from aluminum (work out this problem if you do not see why).

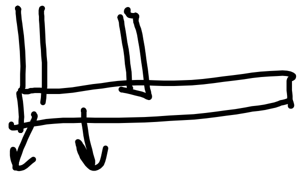


Rod I



Rod II



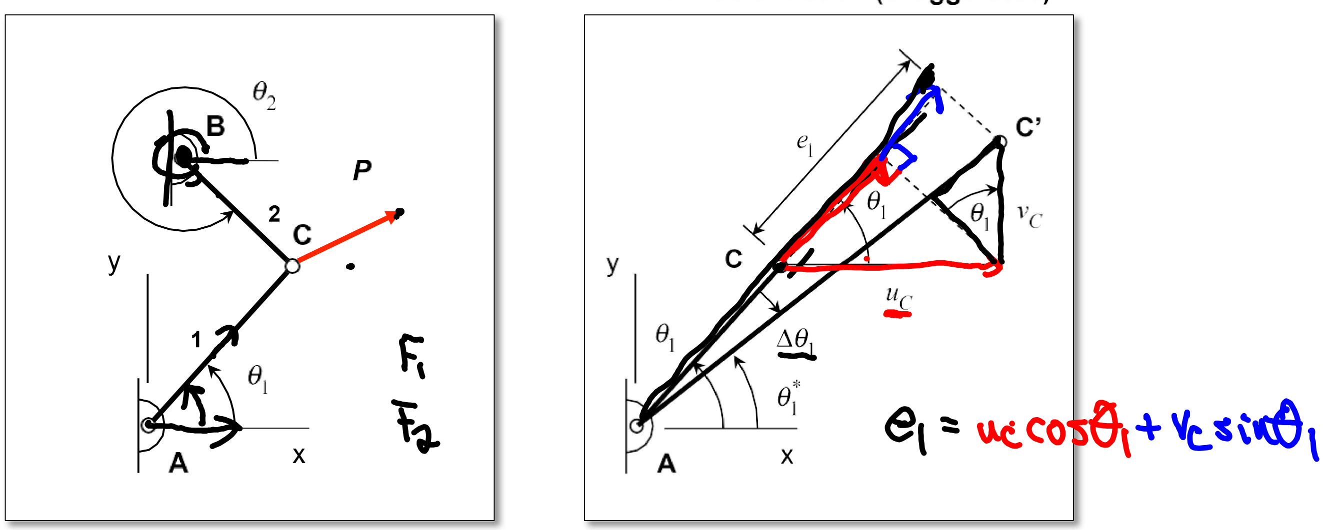


d) Stress and deformation analysis of one-node planar trusses

Up to this point in this course and in your earlier mechanics courses, we have performed force analyses of members in planar trusses using some combination of the method of sections and the method of joints. From these results, we know:

- The component of stress normal to the cross section of a member is found from $\sigma = P / A$ where P is the load carried by the member and A is the cross-sectional area of the member.
- The total elongation of the member is: $e = PL / AE$, where L is the length of the member, E is the Young's modulus of the material and A is the cross sectional area of the member.

BEFORE **deformation** ***AFTER*** **deformation (exaggerated)**



$e_1 = u_C \cos\theta_1 + v_C \sin\theta_1$

Now let's determine the deformation of the members in a truss due to the loading. Consider the simple truss shown above loaded with a force P at joint C. Since this is a determinate truss, we can determine the loadings carried by the two members 1 and 2, F_1 and F_2, respectively, from standard equilibrium analysis. From these, we can calculate the elongations of member 1 and 2 as,

$$e_1 = \frac{F_1 L_1}{EA_1} \tag{7}$$

$$e_2 = \frac{F_2 L_2}{EA_2} \tag{8}$$

respectively. Based on these results, what are the horizontal and vertical components of displacement of the node from C to C' (u_C and v_C, respectively) as a result of this deformation?

To answer this question, first note that the total movement of C takes into account both movement causing strain (the elongation e_1 along axis of member) and movement causing no axial strain (rigid rotation $\Delta\theta_1$ of the member). From the preceding figure, we see that for a small rotation angle $\Delta\theta_1$, the components (u_C, v_C) of the displacement of C that contribute to the elongation of member 1, e_1, are those along the axis of member 1; that is, from the figure we have:

$$e_1 = u_C\, cos\theta_1 + v_C\, sin\theta_1 \tag{9}$$

Similarly, for member 2, we can write:

$$e_2 = u_C\, cos\theta_2 + v_C\, sin\theta_2 \tag{10}$$

Substituting equations (7) and (8) into equations (9) and (10) provides us with the two equations needed to solve for u_C and v_C:

$$\left.\begin{aligned} u_C\, cos\theta_1 + v_C\, sin\theta_1 &= \frac{F_1 L_1}{EA_1} \\ u_C\, cos\theta_2 + v_C\, sin\theta_2 &= \frac{F_2 L_2}{EA_2} \end{aligned}\right] \text{ solve for nodal displacement.}$$

Please note that the above elongation-displacement equations rely on the following angle definitions of the truss elements.

Definition of truss member angles

Let θ_j be the angle of the jth truss member:

- θ_j is measured counterclockwise with respect to the positive x-axis, and
- with the origin of the x-axis placed at the point on the element that is pinned to ground

(as demonstrated by angles θ_1 and θ_2 in the preceding figure).

Only the angles defined as above are valid for the above compatibility equations. Before starting your analysis, you should clearly identify these truss member angles.

As we will see in some of the following examples, we need to use the above deformation analysis in order to do stress analysis of indeterminate trusses.

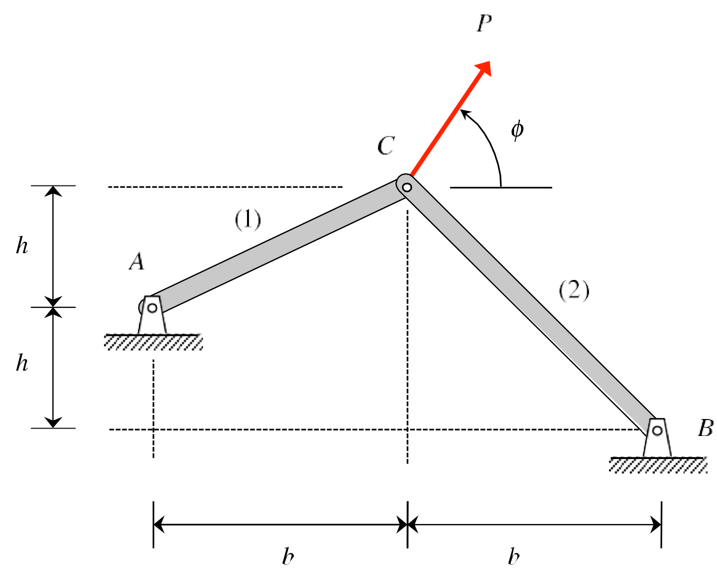
Example 6.10

A two-member truss is shown below with a force P acting on joint C.

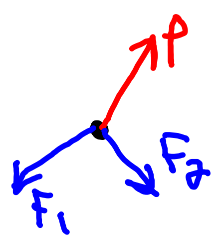
a) Determine the axial stresses in the two members.

b) Determine the horizontal and vertical components of displacement, u_C and v_C, at node C.

Use the following numerical values: $\phi = 36.87°$, $h = 2m$, $b = 3m$, $P = 500kN$, $E_1 = E_2 = 60GPa$ and $A_1 = A_2 = 2000mm^2$.



1.)



$$\left.\begin{matrix}\Sigma F_x \\ \Sigma F_y\end{matrix}\right\} \begin{matrix}\text{2 eqns} \\ \text{2 unknowns.}\end{matrix}$$

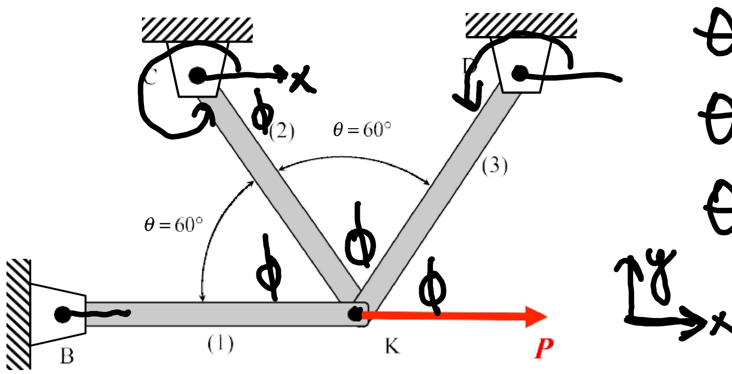
$$\left.\begin{matrix} e_1 = u_C \cos\theta_1 + v_C \sin\theta_1 = \dfrac{F_1 L_1}{E_1 A_1} \\ e_2 = u_C \cos\theta_2 + v_C \sin\theta_2 = \dfrac{F_2 L_2}{E_2 A_2}. \end{matrix}\right\} \begin{matrix}\text{2 eqns} \\ \text{2 unknowns.}\end{matrix}$$

Example 6.12

Each of the three truss members has a length of L and a Young's modulus of E. The cross-sectional areas of the members are $A_1 = A_2 = A_3 = A$. A horizontally-acting force P is applied at joint K.

a) Determine expression for the horizonal and vertical components of the displacement of joint K.

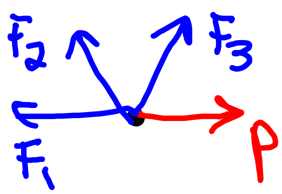
b) Determine the axial forces in the three members of the truss.



$\theta_1 = 0$

$\theta_2 = 300°$

$\theta_3 = 240°$



1.) Equilibrium

$$\left.\begin{aligned} \Sigma F_x &= P + F_3\cos\phi - F_2\cos\phi - F_1 = 0 \\ \Sigma F_y &= F_2\sin\phi + F_3\sin\phi = 0 \end{aligned}\right\}$$

3 unknowns
2 eqns
1

2.) Force-elongation

$$e_1 = \frac{F_1 L_1}{E_1 A_1} \qquad e_2 = \frac{F_2 L_2}{E_2 A_2} \qquad e_3 = \frac{F_3 L_3}{E_3 A_3}$$

+3 unknowns
+3 eqns

3.) Compatibility

$$\left[\begin{array}{l} e_1 = u_K \cos\theta_1 + v_K \sin\theta_1 \\ e_2 = u_K \cos\theta_2 + v_K \sin\theta_2 \\ e_3 = u_K \cos\theta_3 + v_K \sin\theta_3 \end{array}\right.$$

+2 unknowns
+3 eqns.

$\left.\right\}$ 8 unknowns
8 eqns.

$$e_1 = u_K$$

$$e_2 = \frac{1}{2}u_K - \frac{\sqrt{3}}{2}v_K$$

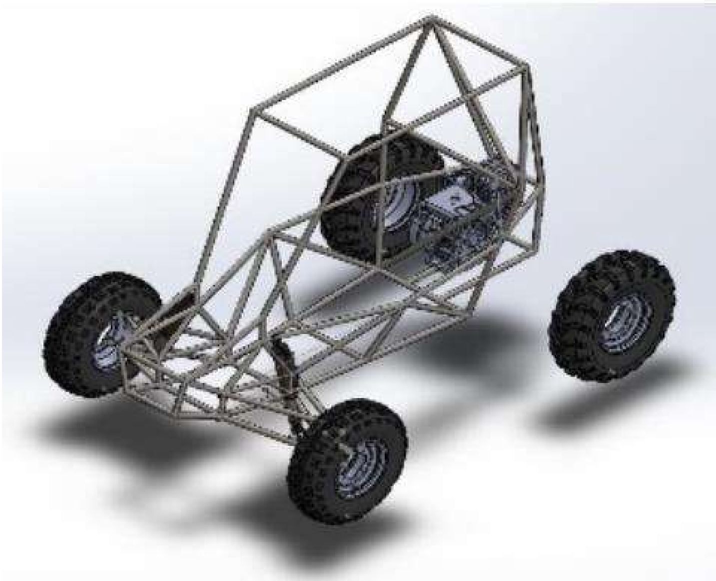
$$e_3 = -\frac{1}{2}u_K - \frac{\sqrt{3}}{2}v_K$$

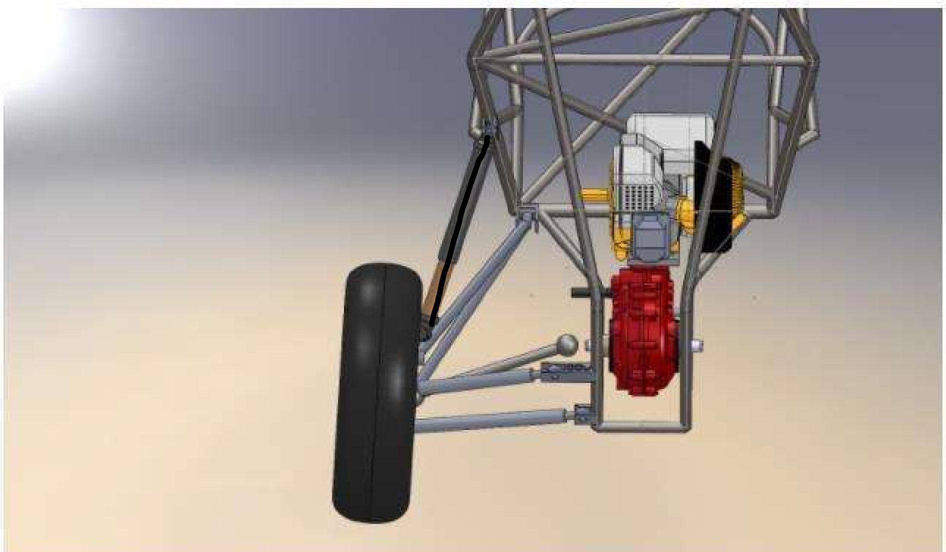
$$\left.\underline{e_2 + e_3} = -\sqrt{3}\,v_K\right]$$

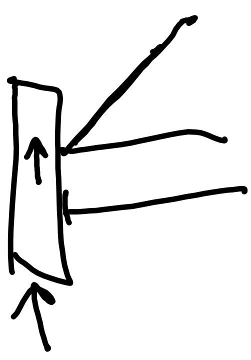
$$e_2 = \frac{1}{2}e_1 + \frac{1}{2}(e_2 + e_3)$$

$$e_2 = e_1 + e_3$$

Planar Truss Application Example







7. Thermal Effects

Objectives:
To introduce the concept of strains developed due to temperature changes in material and how these can lead to thermal stresses for constrained members.

Background:

General 3D stress-strain relationships for isotropic, linearly elastic material experiencing mechanical loads and temperature change ΔT :

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] + \boxed{\alpha \Delta T}$$

$$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right] + \boxed{\alpha \Delta T}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right] + \boxed{\alpha \Delta T}$$

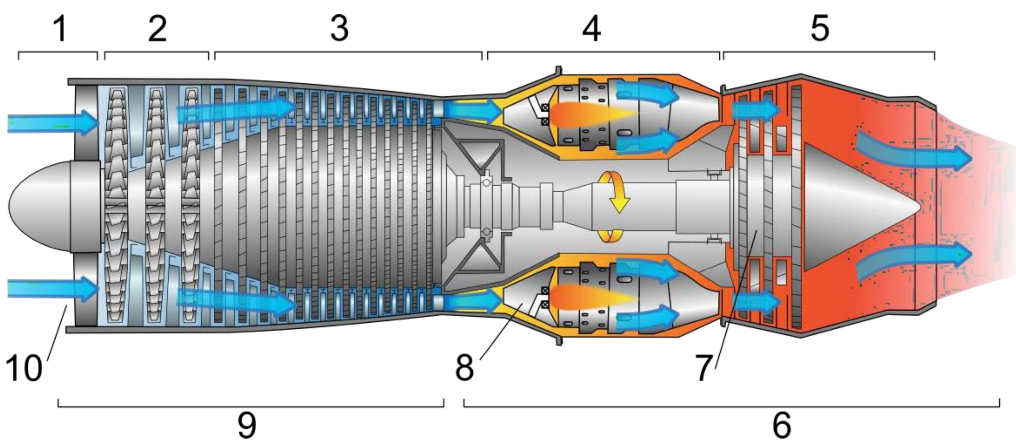
thermal strains

where E, ν and α are the elastic modulus, Poisson's ratio and coefficient of thermal expansion, respectively, for the material.

Thermal Expansion Examples

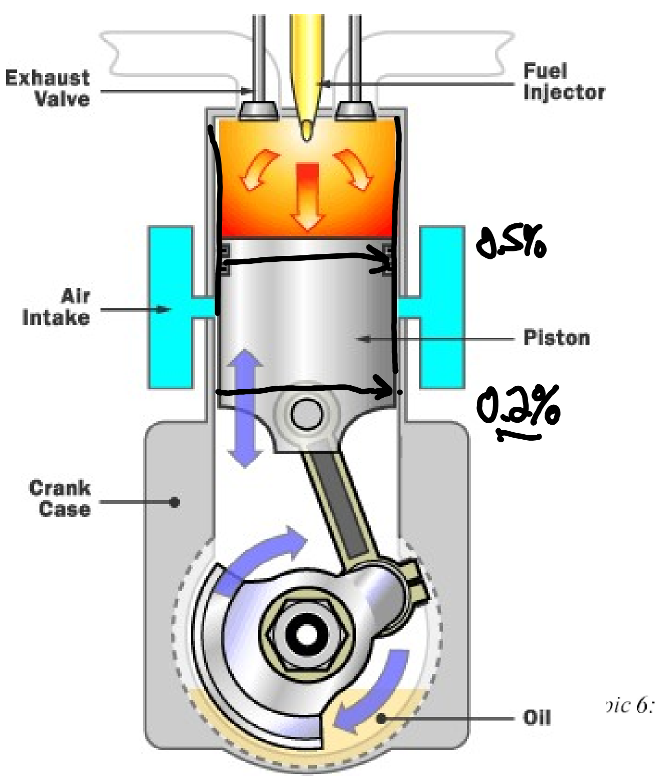
$8\times10^{-6}\,(\text{in}/\text{in})\,{}^\circ F^{-1}$

$= 0.51\ \text{in}/\text{mile}/{}^\circ F$



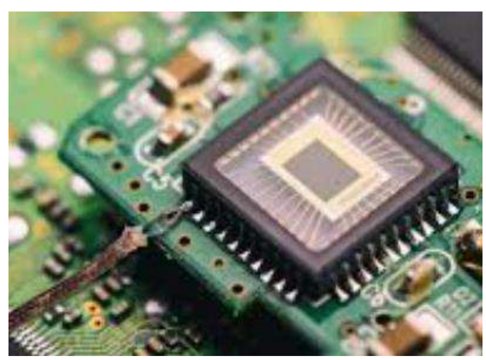
25°C

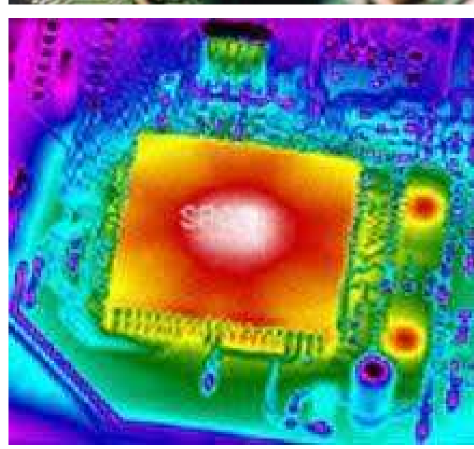
→ 1500°C

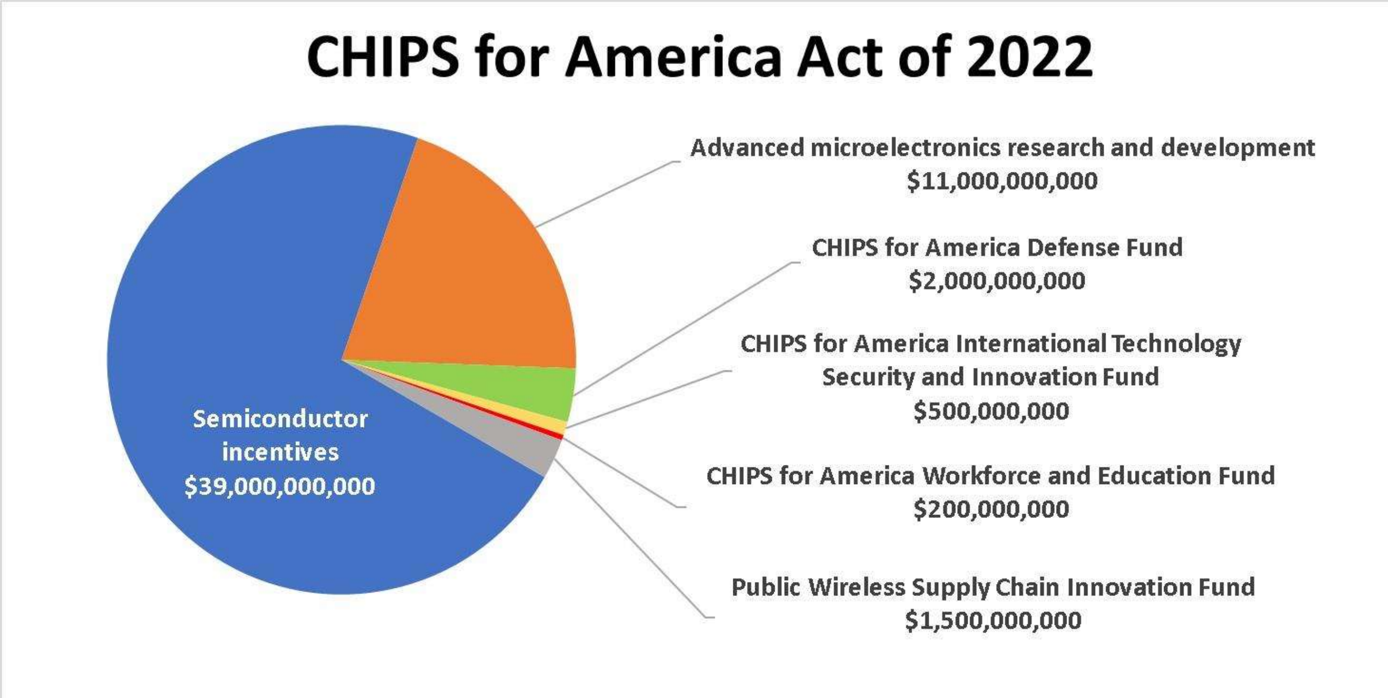


0.5%

0.2%







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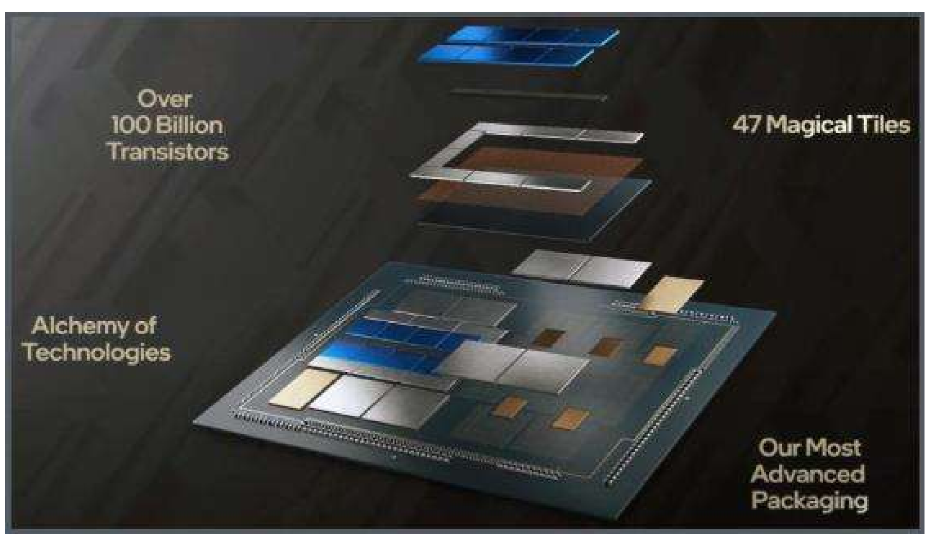


Axial deformation

Lecture Notes

Member elongation

For an axially-loaded member accounting for thermal effects, we have the axial strain given by:

$$\varepsilon = \frac{du}{dx} = \frac{\sigma_x}{E} + \alpha\Delta T = \frac{P}{AE} + \alpha\Delta T$$

Therefore, for a member experiencing a resultant axial load P, length L and cross sectional area A, the elongation in the x-direction is found from integration of the above equation:

$$e = u(L) - u(0) = \int_0^L \left(\frac{P}{EA} + \alpha\Delta T \right) dx$$

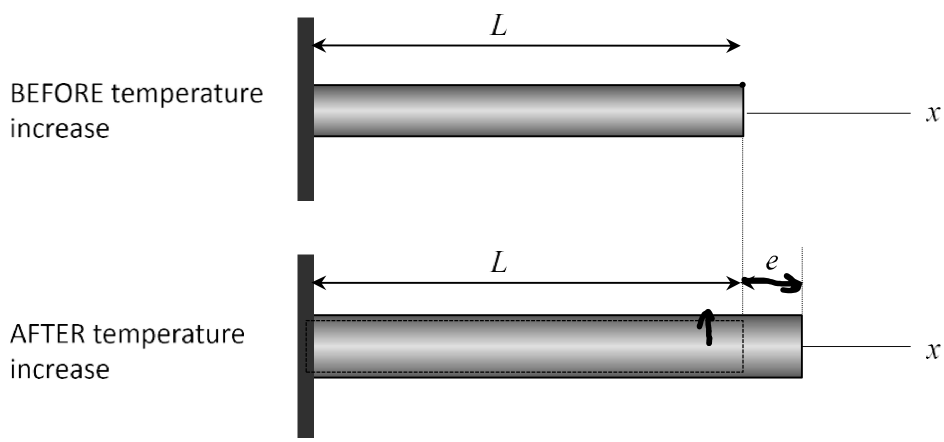
If α, P, E, A and ΔT are constant along the length of the member, then the above integral reduces to:

$$e = \frac{PL}{EA} + \alpha\Delta TL$$

Discussion

Thermal strains can exist in the absence of stress. This is clear from the above stress-strain relations by setting $\sigma_x = 0$; non-zero thermal strain in the x-direction can still remain.

For example, consider a thin rod whose left end is fixed and right end is free. If the material experiences a uniform increase in temperature ΔT, the rod expands uniformly with a total elongation of $e = \alpha\Delta TL$. Since there are no external reactions to resist this expansion, there are no stresses developed in the rod. (To see this, make a cut through the rod at any point along its length, keeping the right hand side. From the FBD of the right side, we see that the axial force, and therefore the axial stress, in the rod is zero.)

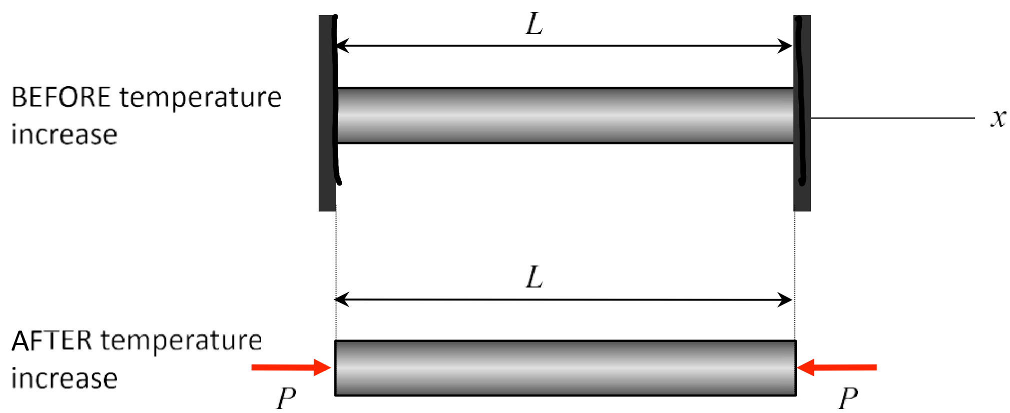


For this case, the rod has non-zero strain and zero stress: $\varepsilon_x \neq 0$ and $\sigma_x = 0$.

Now, reconsider the same member as above except that both ends are fixed, as seen below. As before, if the member is heated the material will expand. However, since the member is prevented from moving past the fixed walls, a compressive axial force P is developed in the member. Since the total elongation of the member remains zero (because of the fixed walls), we see that the compressive force is given by:

$$e = 0 = \frac{PL}{EA} + \alpha \Delta T L \quad \Rightarrow \quad P = -EA\alpha\Delta T$$

producing a compressive stress of $\sigma_x = \dfrac{P}{A} = -E\alpha\Delta T$ Here, the temperature change results in thermal stresses.



For this case, the rod has zero strain and non-zero stress: $\varepsilon_x = 0$ and $\sigma_x \neq 0$.

Reflect back on the results of these two examples. In the first, we found zero stress and non-zero axial strain. In the second, we found zero axial strain and non-zero stress.

In summary, reactive loads are needed to produce thermal stresses from thermal strains. Without these, the member will experience thermal strains, but no thermal stresses.

What temperature change can get steel to break?

$\sigma_Y = 350\ MPa$

E = 200 GPa

$\alpha = 16\text{x}10^{-6}\ ^\circ C^{-1}$

$\sigma_x = -E\alpha\Delta T$

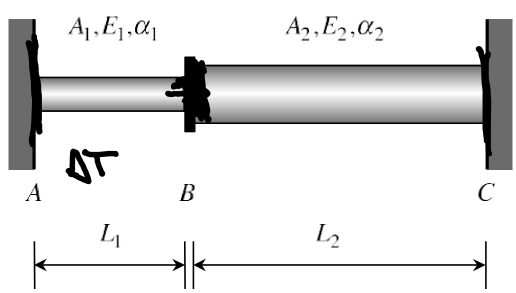
$$\Delta T = \frac{-350\times10^6}{(200\times10^9)(16\times10^{-6})} = 109^\circ C.$$

Example 7.1

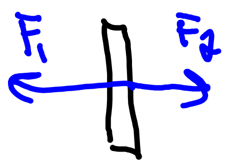
Two uniform, linearly elastic rods are joined together at B, and the resulting two-segment rod is attached to rigid supports at A and C. The rods are initially unstressed.

a) Determine the axial stresses if rod (1) is raised by ΔT and the temperature of rod (2) is held constant.
b) Determine the displacement of node B.

Use the following: $L_1 = L$, $L_2 = 2L$, $E_1 = E_2 = E$, $\alpha_1 = \alpha_2 = \alpha$, $A_1 = A$ and $A_2 = 2A$



$$\boxed{\begin{array}{ll} F_1 = -ve & \text{guess} \\ F_2 = -ve. & \end{array}}$$



1.) $\Sigma F_x = F_2 - F_1 = 0 \qquad F_1 = F_2. \quad \}$

2.) Force-elongations

$$e_1 = \frac{F_1 L_1}{E_1 A_1} + \alpha_1 \Delta T_1 L_2 = \frac{F_1 L}{EA} + \alpha \Delta T L$$

$$e_2 = \frac{F_2 L_2}{E_2 A_2} + \alpha_2 \cancel{\Delta T_2}^{0} L_2 = \frac{F_2 (2L)}{E(2A)} = \frac{F_2 L}{EA}.$$

3.) $e_1 + e_2 = 0$

4.) $e_1 = -e_2$

$$\frac{F_1 L}{EA} + \alpha \Delta T L = -\frac{F_1 L}{EA}.$$

$$e = \frac{FL}{EA}.$$

$$\frac{2F_1 L}{EA} = -\alpha \Delta T L$$

$$F_1 = -\frac{\alpha \Delta T E A}{2}.$$ → compression

$F_1 = F_2$ ⇒ compression.

$$\left.\begin{aligned} \sigma_1 &= \frac{F_1}{A_1} = -\frac{1}{2} E \alpha \Delta T \\ \sigma_2 &= \frac{F_2}{A_2} = -\frac{1}{4} E \alpha \Delta T \end{aligned}\right\}$$ both compressive.

$u_B = e_1$

$u_B = \cancel{u_A}^{0} + e_1$

$$e_1 = -\underbrace{\tfrac{1}{2}\alpha \Delta T E A}_{F_1}\left(\frac{L}{EA}\right) + \alpha \Delta T L.$$

$$u_B = \alpha \Delta T L\left(1 - \tfrac{1}{2}\right) = \tfrac{1}{2}\alpha \Delta T L$$ ⇒ positive.