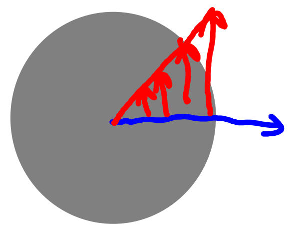
Lecture 10 Review

How does the shear stress on a cross-section depend on the radius?

- It doesn't
- ρ^1
- ρ^2
- ρ^3
- ρ^4

$\tau(\rho)$ $\quad$ $\tau = G\gamma$ $\quad$ $\rho \frac{d\phi}{dx}$.

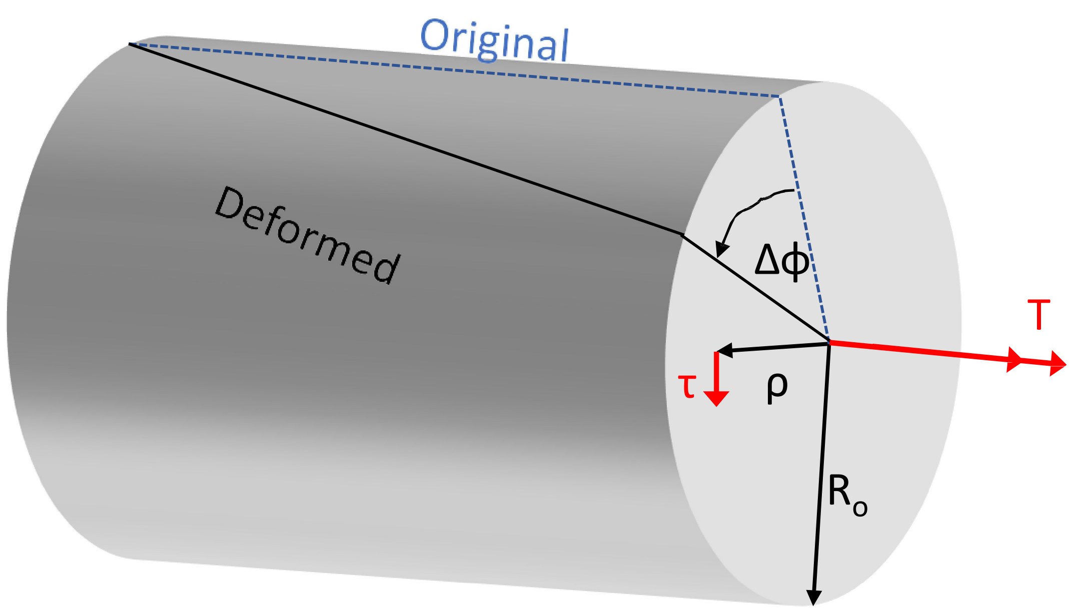


For a solid shaft, how does the polar area moment (I_p) depend on the outer radius?

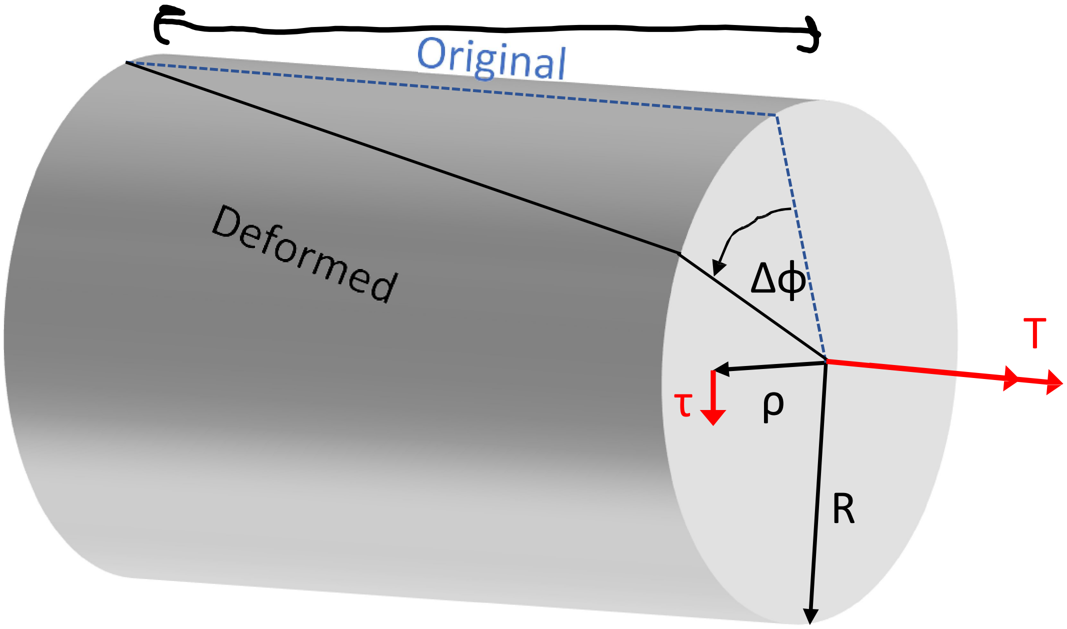
- It doesn't
- R_o^1
- R_o^2
- R_o^3
- R_o^4

$$T = \int \rho\tau \, dA$$

$$T = m \underbrace{\int \rho^2 dA}_{I_p = \frac{\pi}{2} R^4}.$$



Stress analysis of members in torsion

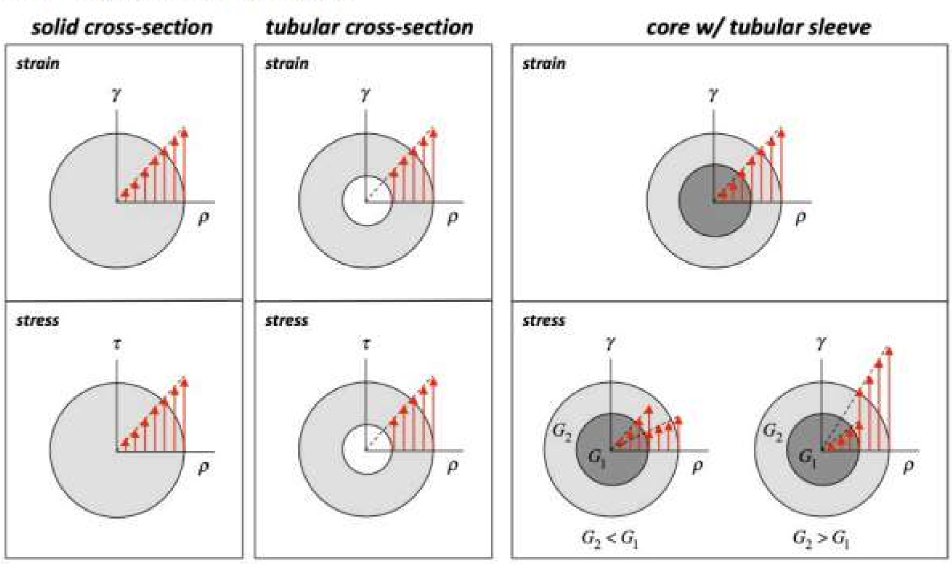


$$\tau = \frac{T\rho}{I_p} \qquad \Delta\Phi = \frac{TL}{GI_p} \qquad I_p = \frac{\pi}{2} r_0^4$$

Summary: torsion stresses in shafts

Consider an axial torque T acting on a shaft with a circular cross section.

- *STRAIN*: The shear strain, γ, varies linearly with radius, ρ, through the cross-section of the shaft, regardless of the material makeup of the cross-section.
- *STRESS*: Across annular regions on the cross-section where the material makeup is a constant, the shear stress, τ, varies linearly with radius, ρ, through the cross-section of the shaft: $\tau = G\gamma = T\rho / I_P$ where I_P is the polar area moment of the cross section.
- *STRAIN/STRESS DISTRIBUTIONS*:



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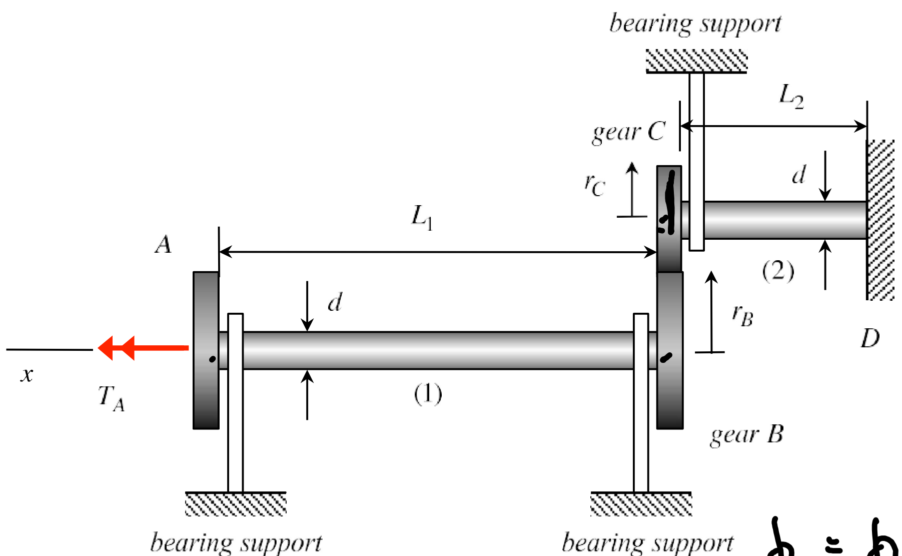
Example 8.4

A torque of 400 N-m is applied to gear A of a two-shaft system and is transmitted through gears B and C to a fixed end D. The shafts are made of a material having a shear modulus of G, and each shaft has a diameter of $d = 32\ mm$. The shafts are supported by frictionless bearings.

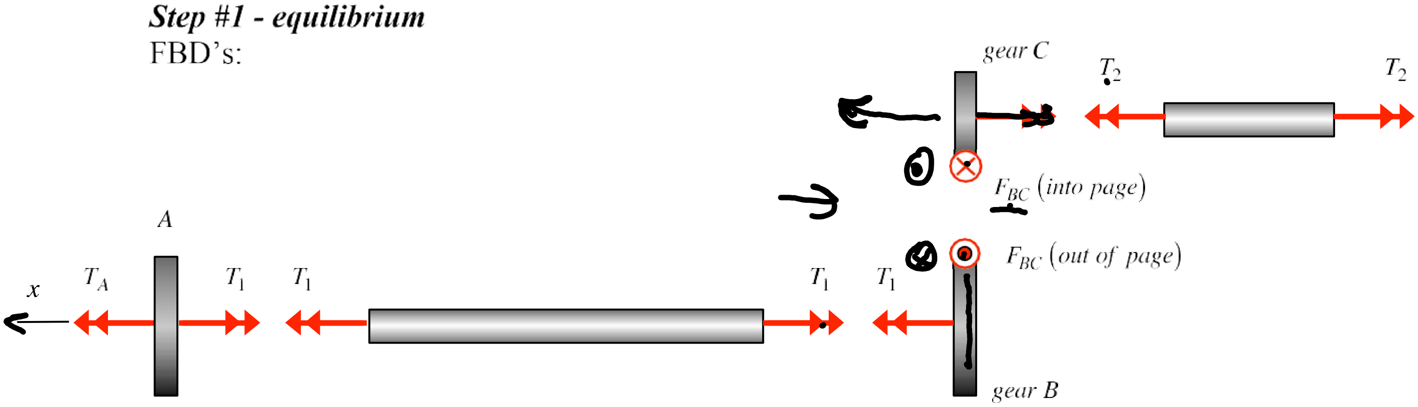
a) Determine the maximum shear stress in each shaft.

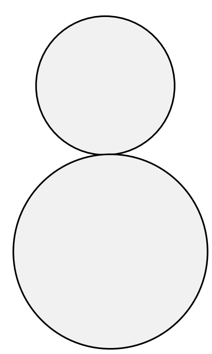
b) Determine the angular rotation of gear A relative to its unloaded position.



$\phi_C = \phi_D + \Delta\phi_2.$

Solution
Step #1 - equilibrium
FBD's:





FBD of 1: $\sum T_x = T_A - T_1 = 0 \quad \Rightarrow \quad T_1 = T_A$ (1)

FBD of gear B: $\sum T_x = T_1 - r_B F_{BC} = 0 \quad \Rightarrow \quad F_{BC} = T_1 / r_B$ (2)

FBD of gear C: $\sum T_x = -r_C F_{BC} - T_2 = 0 \quad \Rightarrow \quad F_{BC} = -T_2 / r_C$ (3)

Equating (2) and (3): $F_{BC} = T_1 / r_B = -T_2 / r_C \quad \Rightarrow \quad T_2 = -r_C T_A / r_B$ (4)

Step #2 – Torque-twist equations

$$\Delta\phi_1 = \frac{T_1 L_1}{G I_P} = \frac{T_A L_1}{G I_P} \tag{5}$$

$$\Delta\phi_2 = \frac{T_2 L_2}{G I_P} = -\frac{r_C}{r_B}\frac{T_A L_2}{G I_P} \tag{6}$$

Step #3 – Compatibility equations

$$\phi_C = \Delta\phi_2 \tag{7}$$

$$\phi_B = -\left(r_C / r_B\right)\phi_C = -\left(r_C / r_B\right)\Delta\phi_2 \tag{8}$$

$$\phi_A = \phi_B + \Delta\phi_1 = \Delta\phi_1 - \left(r_C / r_B\right)\Delta\phi_2 \tag{9}$$

Step #4 – Solve

a) Stresses:

$$\left|\tau_1\right|_{\max} = \frac{T_1\left(d/2\right)}{I_P} = \frac{T_A d}{2 I_P} \qquad \left|\tau_2\right|_{\max} = \frac{T_2\left(d/2\right)}{I_P} = \left(\frac{r_C}{r_B}\right)\frac{T_A d}{2 I_P}$$

b) Rotation at A:

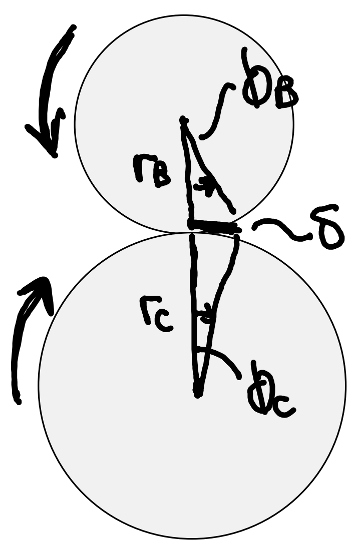
Combining torque-twist equations (5)-(6) with compatibility equation (9):

$$\phi_A = \frac{T_A L_1}{G I_P} + \left(\frac{r_C}{r_B}\right)^2 \frac{T_A L_2}{G I_P} = \left[1 + \frac{L_2}{L_1}\left(\frac{r_C}{r_B}\right)^2\right]\frac{T_A L_1}{G I_P}$$

$$\tan\phi_B \approx \phi_B = \frac{\delta}{r_B}$$

$$\phi_C = \frac{\delta}{r_C}.$$

$$\phi_B = -\frac{r_C}{r_B}\phi_C$$



c) Statically indeterminate shafts with externally applied torques

As we have seen above, the ability to determine the maximum shear stress in a shaft depends on our ability to determine the resultant torque T at a given cross section. Like axially-loaded members, many torsional problems are statically indeterminate. Recall that statically-indeterminate systems are ones for which we cannot determine internal reactions (torques) through rigid body analysis.

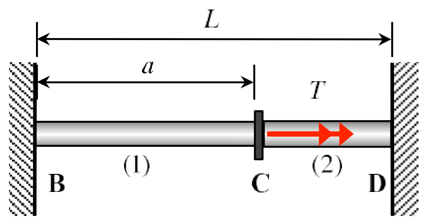
Motivating Example

A torque load T acts at point C on a circular shaft. Determine the reaction torques at the fixed ends B and D. The shaft has a cross-sectional area of A with a shear modulus of G. Answer the following questions:

Q1: Why is this problem indeterminate if considering the shaft as a rigid member?

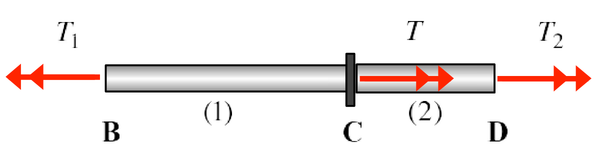
Q2: How does a consideration of strain (deformation) allow you to solve for the reactions?

Q3: Which end (B or D) carries the largest reaction torque if $a > L/2$? Defend your answer with a physical argument.

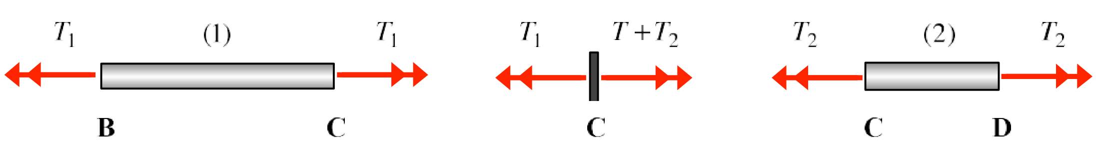


Solution

FBD of entire shaft:



FBD's of sections (1) and (2):



Example 8.5

A stepped shaft AC (made of material with shear modulus G) is subjected to an external torque of T_B at B and is fixed to rigid supports at ends A and C, as shown in the figure below.

a) Determine the torques T_1 and T_2 carried by segments (1) and (2), respectively.

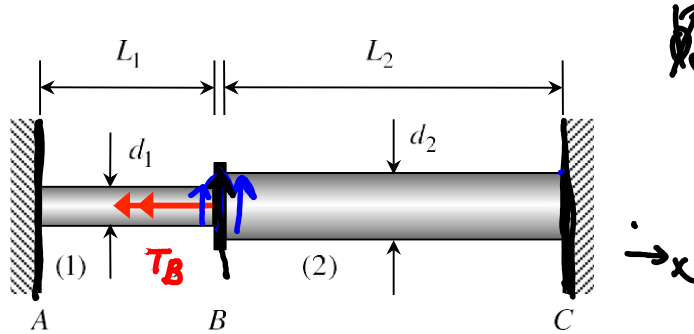
b) Determine the maximum shear stress in each segment.

c) Determine the angle of rotation ϕ_B at joint B.

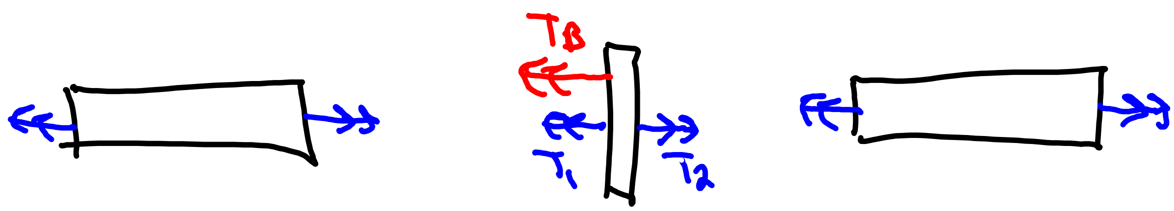
$$\boxed{\phi_B = \cancel{\phi_A}^{0} + \Delta\phi_1}$$

$$\cancel{\phi_C}^{0} = \phi_B + \Delta\phi_2.$$

$$\phi_B = \cancel{\phi_A}^{0} + \Delta\phi_1$$



$$0 = \Delta\phi_1 + \Delta\phi_2.$$



1.) $(\Sigma M)_B = T_2 - T_1 - T_B = 0$ } 2 unknowns, 1 eq/n

2.) $\Delta\phi_1 = \dfrac{T_1 L_1}{G_1 I_p}$ $\Delta\phi_2 = \dfrac{T_2 L_2}{G_2 I_{p2}}$ $I_{p1} = \frac{\pi}{2}(d_1/2)^4 = \frac{\pi}{32} d_1^4$

$I_{p2} = \frac{\pi}{32} d_2^4$

3.) <u>Compatibility</u>

$$\Delta\phi_1 + \Delta\phi_2 = 0$$

4.)

$$\frac{T_1 L_1}{G_1 I_{p1}} + \frac{T_2 L_2}{G_2 I_{p2}} = 0$$

$$T_1 = -\left(\frac{L_2}{L_1}\frac{I_{p1}}{I_{p2}}\right) T_2.$$

$$T_2 - T_1 - T_B = 0$$

$$T_2 + \left(\frac{L_2}{L_1}\frac{I_{p1}}{I_{p2}}\right) T_2 = T_B$$

$$T_2 = \frac{T_B}{1 + \left(\frac{L_2}{L_1}\right)\left(\frac{I_{p1}}{I_{p2}}\right)}$$

b) $\tau_{max} = \dfrac{TR}{I_p}$. $\quad \tau_{max,1} = \dfrac{T_1 (d_1/2)}{\frac{\pi}{2}(d_1/2)^4}$

c) $\phi_B = \Delta\phi_1 = \dfrac{T_1 L_1}{G_1 I_{p1}}$

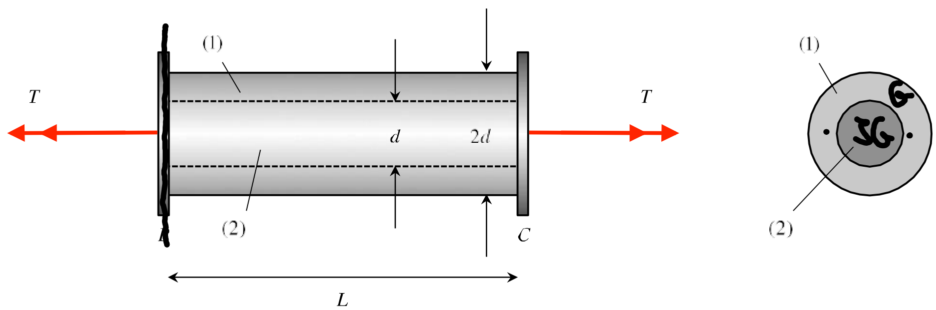
Example 8.7

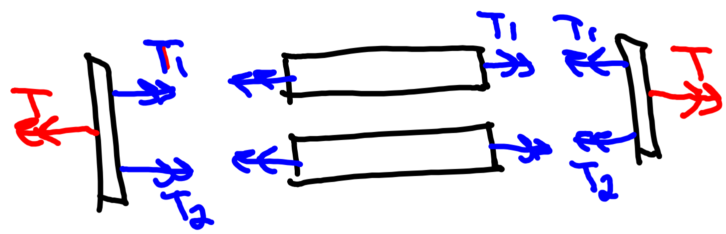
A bimetallic torsion bar consists of a shell (1) and a core (2). The shear moduli for the material making up the shell and core are known to be G and $3G$, respectively. The bar is loaded with an axial torque.

a) Determine the maximum shear stress in the core (2) and the maximum shear stress in the shell (1).

b) Make a sketch of the shear stress distribution across the bar cross section.

c) Determine the total twist angle of the composite bar.





1) Equilibrium

$$(\Sigma M)_C = T - T_1 - T_2 = 0$$

2 unknowns
1 eqn

2.) $$\Delta\phi_1 = \frac{T_1 L}{G I_{p1}}$$

$$I_{p1} = \frac{\pi}{2}\left[\left(\frac{2d}{2}\right)^4 - \left(\frac{d}{2}\right)^4\right] = \frac{\pi}{2}d^4 - \frac{\pi}{32}d^4 = \frac{15\pi}{32}d^4$$

$$\Delta\phi_2 = \frac{T_2 L}{3G I_{p2}}$$

$$I_{p2} = \frac{\pi}{2}\left(\frac{d}{2}\right)^4 = \frac{\pi}{32}d^4$$

3.) $\Delta\phi_1 = \Delta\phi_2$

4.) $\frac{T_1 L}{G I_{p1}} = \frac{T_2 L}{3G I_{p2}}$

$T_2 = 3\left(\frac{I_{p2}}{I_{p1}}\right) T_1$

$T - T_1 - T_2 = 0$

$T_1 + 3\left(\frac{I_{p2}}{I_{p1}}\right) T_1 = T$

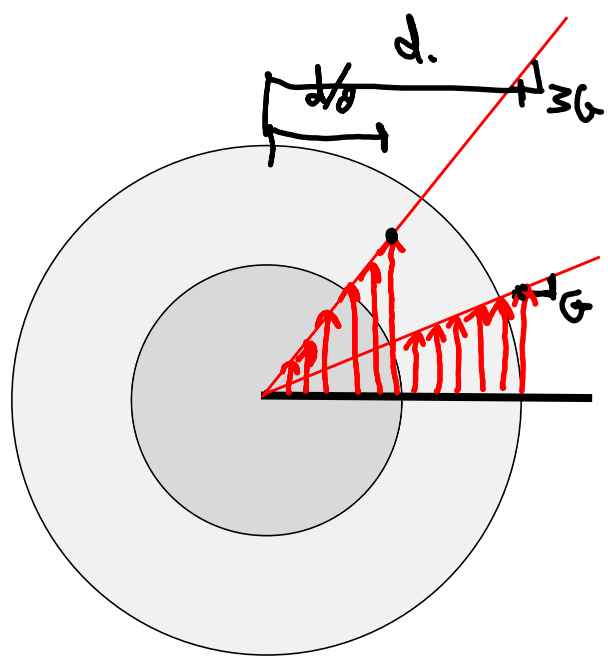
$$T_1 = \frac{T}{1 + 3\left(\frac{I_{p2}}{I_{p1}}\right)} = \frac{T}{1 + 3\left(\frac{1}{15}\right)} = \frac{T}{1 + \frac{3}{15}} = \frac{T}{\frac{15}{15} + \frac{3}{15}} = \boxed{\left(\frac{15}{18}\right) T}$$

$T_2 = \frac{3}{18} T$

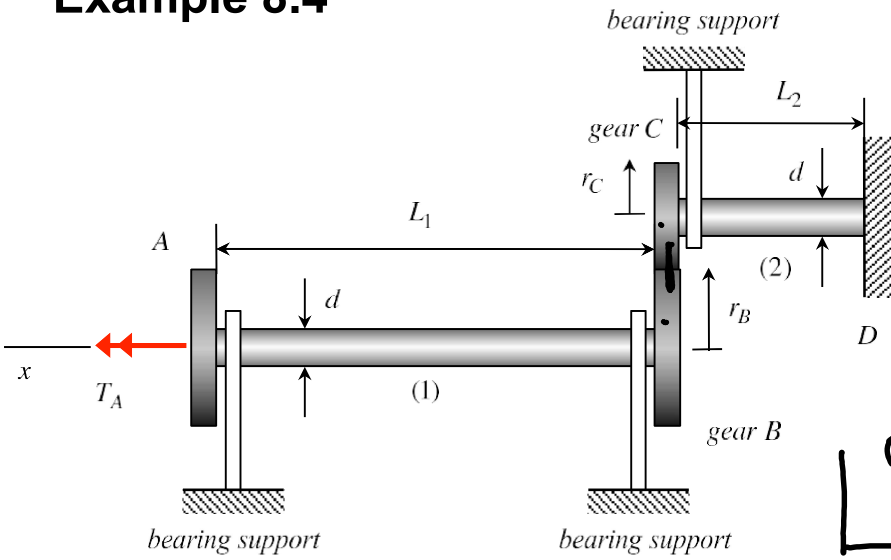
$\tau = \frac{T\rho}{I_p}$

$\tau = G\left(\frac{\Delta\phi}{L}\right) R$

$\tau = G\rho \frac{d\phi}{dx}$

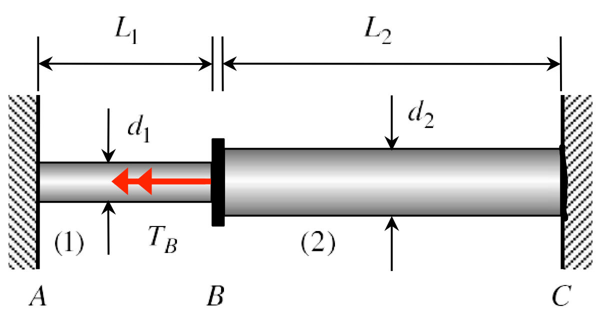


Example 8.4



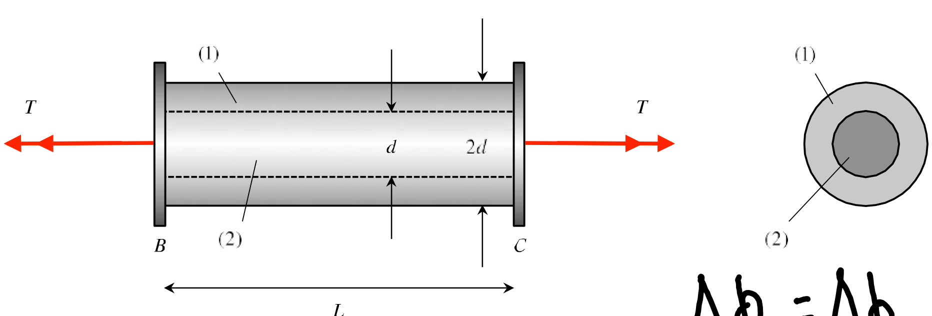
$$\phi_C = -\frac{r_B}{r_C}\phi_B$$

Example 8.5



$$\Delta\phi_1 + \Delta\phi_2 = 0$$

Example 8.7



$$\Delta\phi_1 = \Delta\phi_2$$

Stress analysis of members in torsion