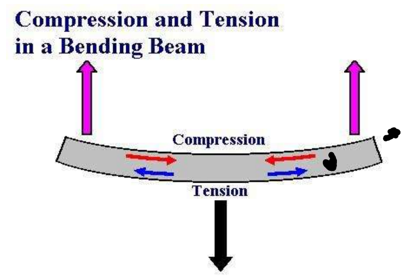
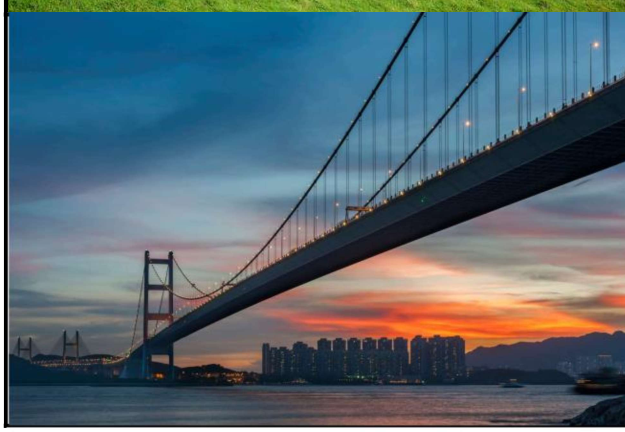
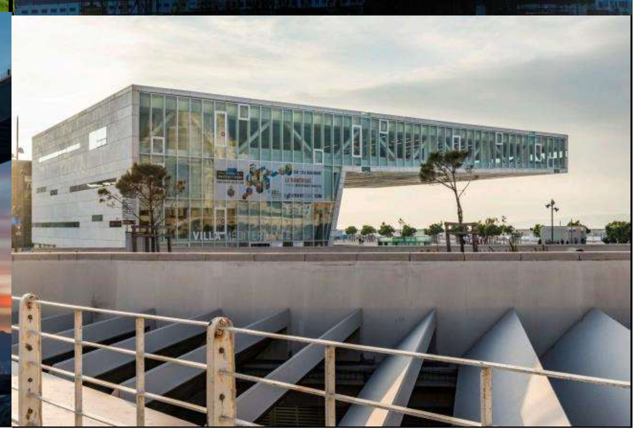
Chapters 9-11: Beams

Applications

Beams are structural members that are designed to support transverse loads, that is, loads that are perpendicular to the longitudinal axis of the beam. A beam resists the applied loads by a combination internal transverse shear force and bending moment.





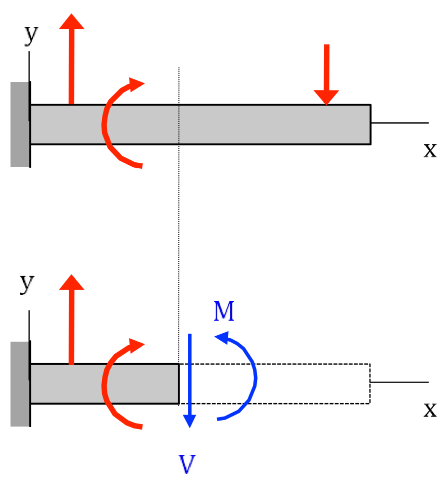


10. Beams: Flexural and shear stresses

Objectives:
To develop relationships for the normal stresses and shear stresses corresponding to the internal bending moment and shear force resultants in beams.

Background:

- The bending moment M and shear force V at a cut through the cross section of a beam are couple and force *resultants* of the *normal* and *shear* stresses, respectively, at the cross section.



- Shear force/bending moment equation:

$$V = \frac{dM}{dx}$$

- Axial stress/strain relation:

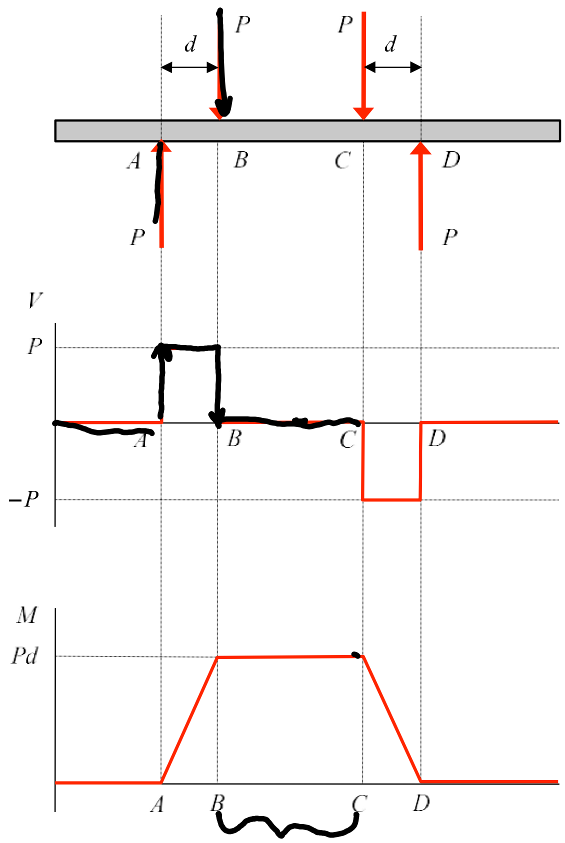
$$\sigma_x = E\varepsilon_x$$

Lecture topics:

a) Strains for pure bending in beams
b) Flexural stresses due to bending in beams
c) Stresses due general transverse force and bending-couple loading of beams

Lecture Notes

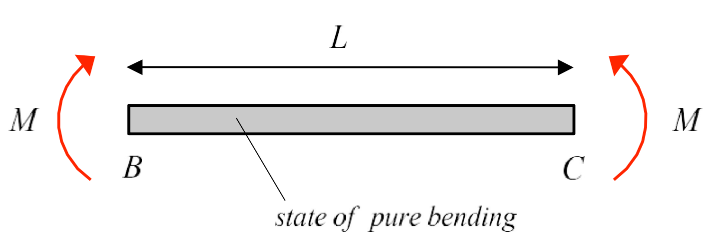
Suppose we consider an example of a beam acted upon by two force/couple pairs resulting from equal magnitude forces P at locations A, B, C and D.



As seen in the above shear-force/bending-moment diagrams:

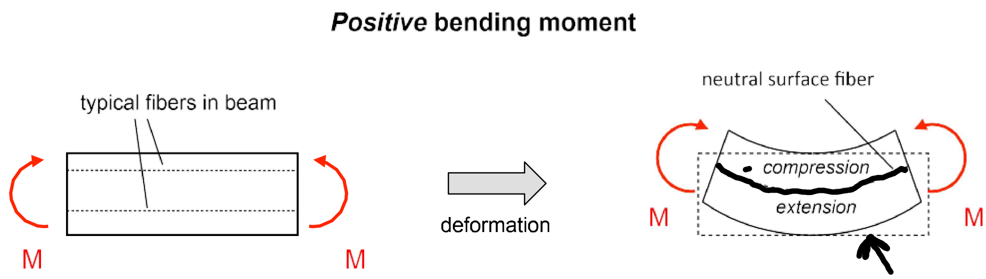
- The shear force in the beam between B and C is zero.
- The bending moment between B and C is a constant value of $M = Pd$.

Therefore, a state of "pure bending" (zero shear force) exists between B and C in the beam. So long as we keep our focus on the section BC of the beam, we can represent the above loading as a beam with equal and opposite couples $M = Pd$ applied at its ends, as shown below.

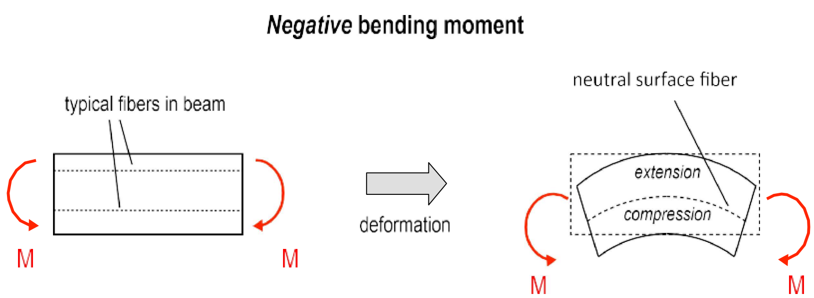


a) Strains for pure bending in beams

In order to view the beam deformations, it is convenient to imagine the beam to be made up of longitudinal fibers parallel to the longitudinal axis of the beam. Under the action of equal and opposite positive bending couples at its ends, the top fibers of the beam will shorten and the bottom fibers of the beam will stretch, as indicated below. The fiber that divides the region of compression from the region of stretch is said to lie on the "*neutral surface*" of the beam.



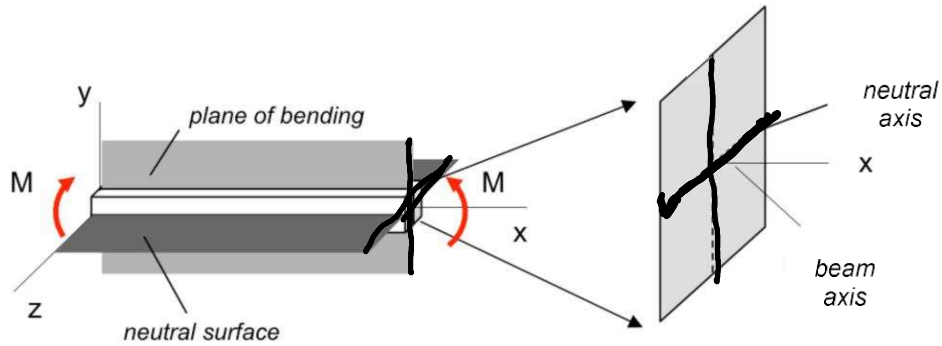
Conversely, under the action of equal and opposite negative bending couples at its ends, the top fibers of the beam stretch and the bottom fibers will shorten.



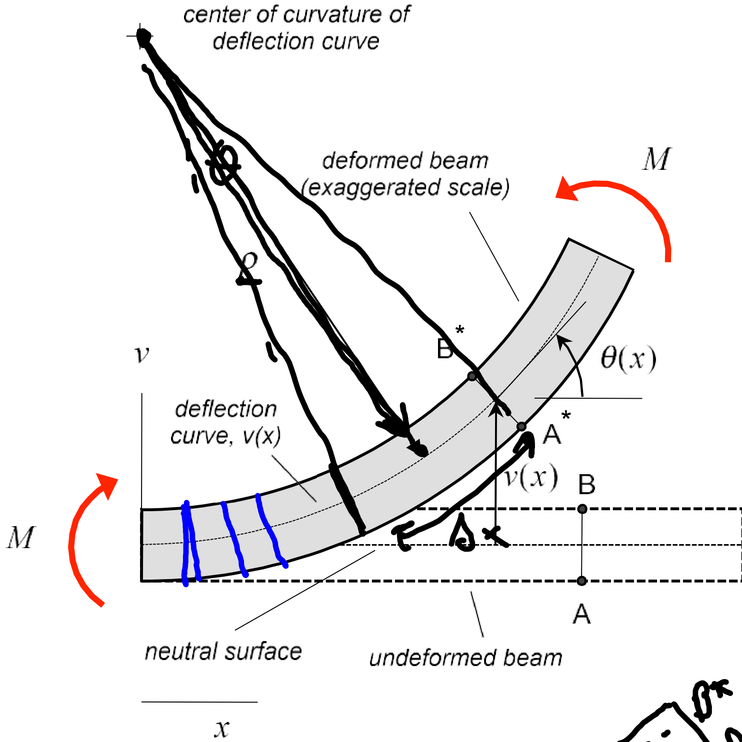
Euler- Bernoulli definitions and kinematic assumptions for thin beams

Consider the following assumptions related to the geometry and loading of a beam:

- The beam has a plane of longitudinal plane of symmetry (xy-plane as shown in following figure) called the "*plane of bending*". Loading and supports for the beam are assumed to be symmetrical about the plane of bending.
- The beam has a longitudinal plane (xz-plane as shown in following figure) perpendicular to the plane of bending on which there is zero longitudinal strain called the "*neutral surface*". The intersection of the neutral surface with the plane of the cross section is called the "*neutral axis*" for the cross section. In the following discussions, it will be assumed that the z-axis will be aligned with the neutral axis of the beam in its undeformed state. The intersection of the plane of bending and neutral surface is known as the beam axis. The deformation of the initially-straight beam axis is known as the "*deflection curve*" of the beam.



- Planar cross sections that are perpendicular to the beam axis before the beam deforms remain perpendicular to the beam axis after deformation. In the following figure are shown two points A and B on a cut made perpendicular to the neutral axis of the undeformed beam. As a result of the application of the bending moment M, cut A-B rotates in the counter-clockwise sense to produce $A^* - B^*$; however, as a result of this assumption, $A^* - B^*$ remains perpendicular to the deflection curve. Also, the radius of curvature of the deflection curve is denoted as ρ in the figure.



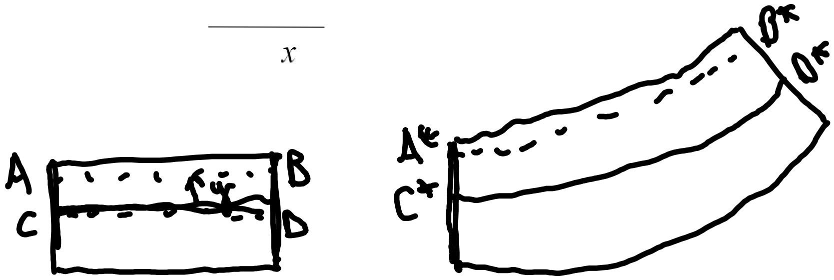
$$L = \theta\rho$$

$$\varepsilon = \frac{A^*B^* - AB}{AB}$$

$$AB = C^*D^* = \Delta x = \theta\rho$$

$$A^*B^* = \theta(\rho - y)$$

$$\varepsilon(y) = \frac{\theta(\rho - y) - \theta\rho}{\theta\rho} = \frac{-y}{\rho}$$



b) Flexural stresses due to bending in beams

Consequences of the Euler-Bernoulli assumptions:

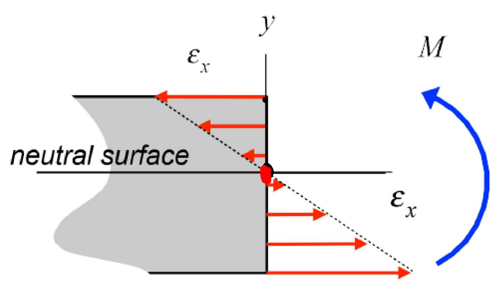
- As a result of the above Euler-Bernoulli assumptions, it can be shown that the axial strain ε_x across a perpendicular cut in the beam has the following distribution in y:

$$\varepsilon_x = -\frac{y}{\rho} \qquad y = \text{distance from neutral axis.} \tag{1}$$

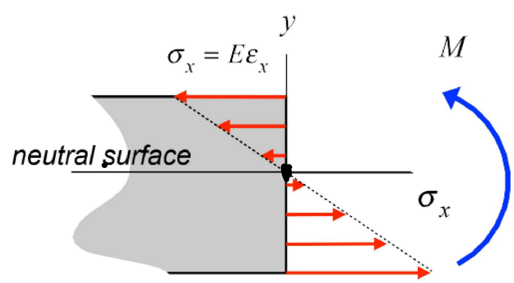
where y is measured from the neutral surface of the beam and ρ is the radius of curvature of the deflection curve for the loaded beam.

- For a linearly-elastic material for the beam, the normal stress distribution in y is therefore:

$$\sigma_x = E\varepsilon_x = -\frac{Ey}{\rho}. \tag{2}$$



strain distribution across cut



stress distribution across cut

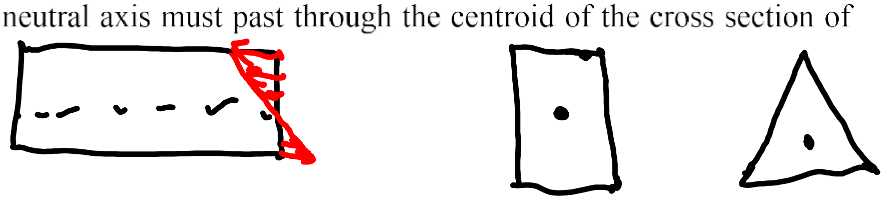
- The resultant axial force on the face of the cut is found by:

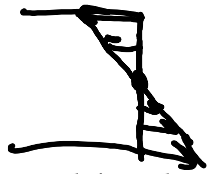
$$0 = F_N = \int_A \sigma_x \, dA = -\frac{E}{\rho}\int_A y \, dA = -\frac{E\bar{y}A}{\rho}$$

where A is the area of the cross section at the cut and $\bar{y}$ is y-position of the centroid of the cut. Since the beam is known to be in pure bending, the resultant axial force on the face of the cut must be zero Therefore, using the above, we see that:

$$\bar{y} = 0 \tag{3}$$

or, in words, the neutral axis must past through the centroid of the cross section of the cut.





RESULT: When studying the stress distribution in beams, determine first the location of the centroid of the cross section – the neutral axis passes through this point.

- The resultant moment about the neutral axis must be equal to the couple M. Therefore,

$$M = -\int_A \sigma_x y \, dA = \frac{E}{\rho} \int_A y^2 \, dA = \frac{EI}{\rho} \tag{4}$$

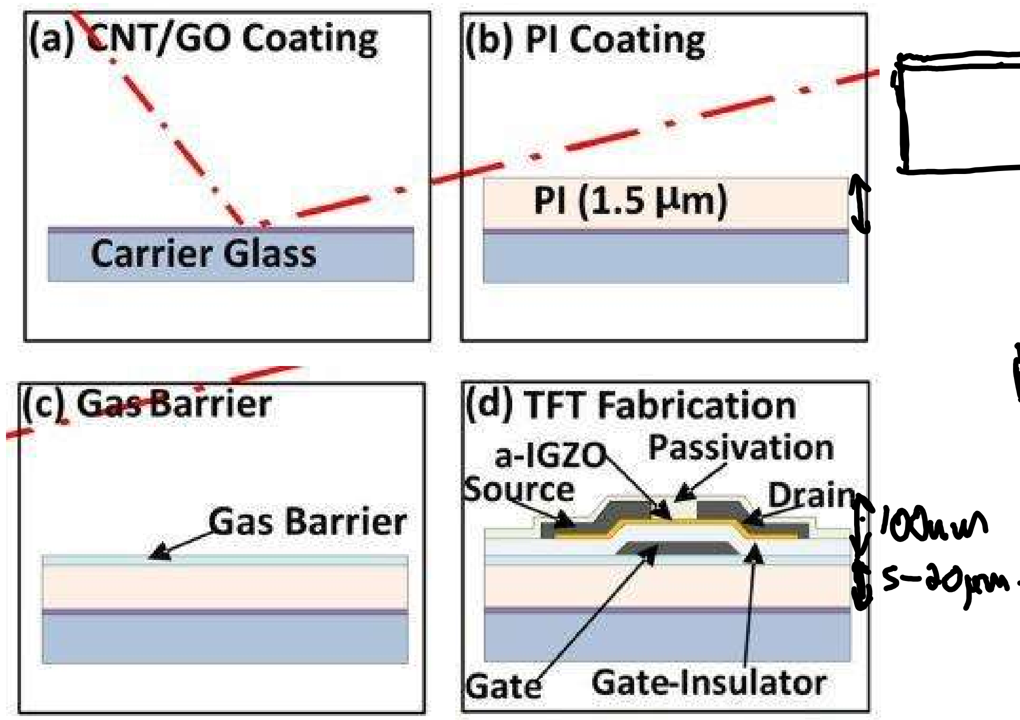
where:

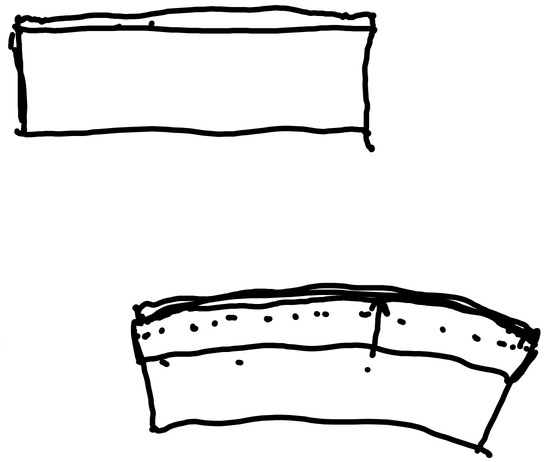
$$\boxed{I = \int_A y^2 \, dA} = \textit{second area moment of cross section} \tag{5}$$

- Combining equations (2) and (4) gives the desired relationship between the applied couple M and the distribution of normal stress across a cross section of the beam:

$$\boxed{\sigma_x = -\frac{M y}{I}} \tag{6}$$

Flexible Electronics

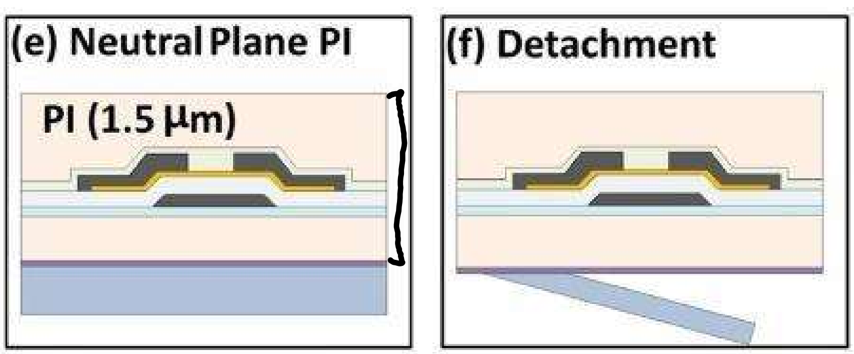


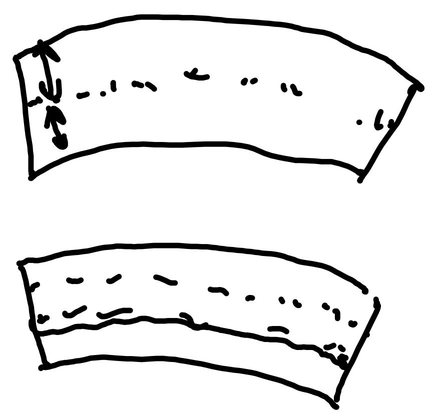


$\varepsilon \propto y$

$\sigma \propto y$

How can we reduce the stress and strain on the active electronics in the device?





Kim et al, Sci Rep, 6:25734, 2016

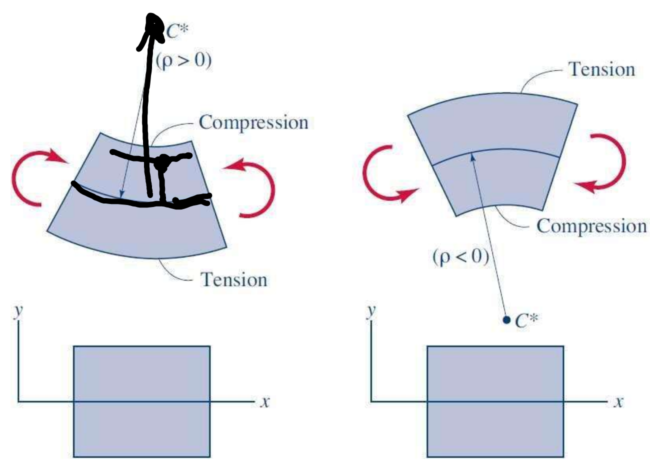
Summary: pure bending at a beam cross section

At a cut through a section of a beam experiencing pure bending (zero shear force, $V = 0$) and abiding by the Euler-Bernoulli assumptions, we can make the following observations (see following figure):

a) Even though loads are applied transverse to the beam, axial strains and stresses are produced. Only normal stresses σ_x exist at the cut. M

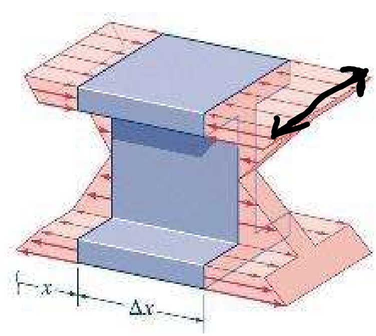
b) The extensional strain $\varepsilon_x = -y/\rho$ is inversely proportional to the radius of curvature of the beam deflection curve at a cross section, x.

c) The signs of ρ and y govern the sign of ε_x. If ρ is positive, the center of curvature of the beam deflection lies above the beam, that is, on the $+y$ side of the beam and the deformed beam is concave upward. Because of the negative sign in the equation of ε_x, the sections above the neutral surface are in compression, while the sections below the neutral surface are in tension.



(*a*) Positive curvature. (*b*) Negative curvature.

$\varepsilon = -\frac{y}{\rho}$



d) The axial strain is not uniform across the section but varies according the height of the point from the neutral axis. Flexural strain reaches maximum at the top and bottom of the beam and is zero at the neutral axis where there is no axial strain.

e) The neutral axis of the cross section (axis of zero strain) passes through the centroid of the cross section.

f) The normal stresses vary linearly in the y-direction: $\sigma_x(y) = -My/I$, where I is the second area moment of the cross section at the cut about the neutral axis. The negative sign in this equation results from sign conventions established earlier. For example, a positive bending moment results in negative (compressive) stress above the neutral axis and positive (tensile) stress below the neutral axis.

g) The normal stresses are constant in the z-direction (into the depth of the beam).

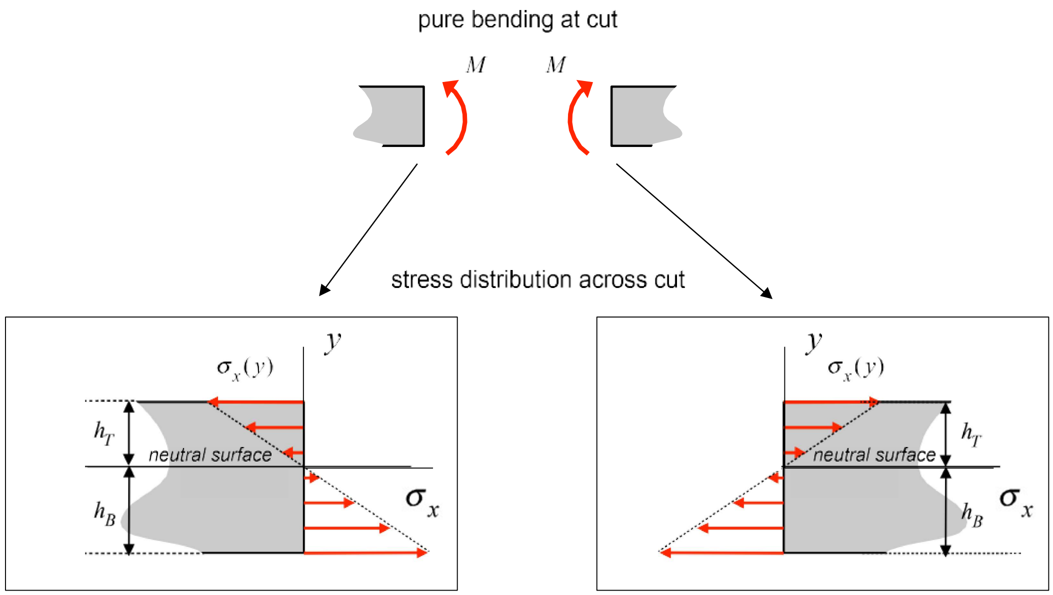
h) The normal stress is zero at the neutral axis.

i) The maximum (magnitude) normal stress exists at the most outer surface of the beam (as measured from the neutral axis). In particular,

$$\left|\sigma_x\right|_{max} = \frac{|M|\,|y|_{max}}{I}$$

where $|y|_{max} = max(h_T, h_B)$.

j) The bending moment M can be written in terms of the radius of curvature ρ of the beam deflection as: $M = EI/\rho$. Since M is a constant over the section of pure bending, the radius of curvature is also a constant. Hence, we conclude that *a section of pure bending of a beam takes on the shape of a circle* (circle = curve of constant radius of curvature).



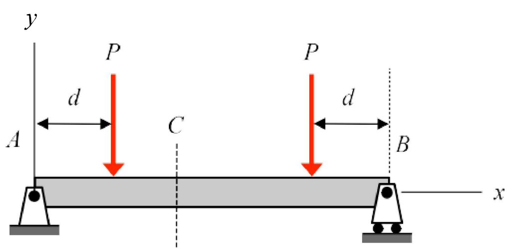
Example 10.1

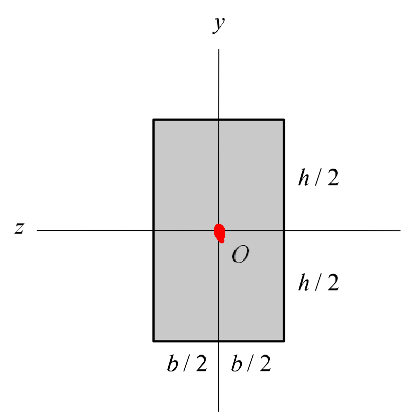
A simply-supported beam is loaded as shown. The cross section at location C of the beam is as shown below right, where C is somewhere between the two applied loads P. Point O on the cross section is on the neutral axis of the beam.

a) Determine the second area of moment of the beam cross section. Leave your answer in terms of b and h.

b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) of the normal stress on the cross section.

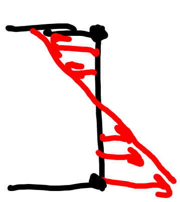




a) $$I = \int_A y^2\,dA = \int_{-b/2}^{b/2}\int_{-h/2}^{h/2} y^2\,dy\,dz$$

$$= \int_{-b/2}^{b/2}\left[\frac{y^3}{3}\right]_{-h/2}^{h/2} dz = \int_{-b/2}^{b/2}\frac{1}{3}\left[\frac{h^3}{8}+\frac{h^3}{8}\right]dz$$

$$= \int_{-b/2}^{b/2}\frac{h^3}{12}\,dz = \frac{bh^3}{12} = I$$

c) 

$$\sigma_{max} = \frac{M\left(\frac{h}{2}\right)}{I} = \frac{Pdh}{2\left(\frac{bh^3}{12}\right)} = \frac{6Pd}{bh^2}.$$

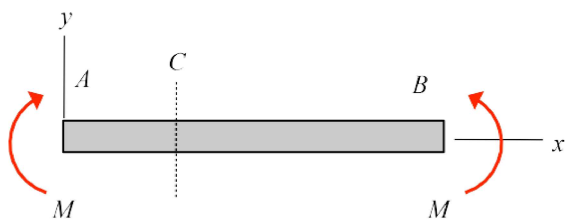
Example 10.2

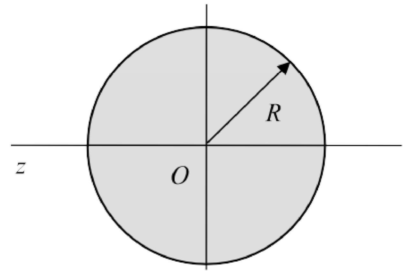
A beam is loaded in pure bending, as shown. The cross section at location C of the beam is as shown below right, where C is somewhere along the length of the beam. Point O on the cross section is on the neutral axis of the beam.

a) Determine the second area of moment of the beam cross section. Leave your answer in terms of R.

b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) of the normal stress on the cross section.





$$I = \int_A y^2 \, dA = \int_0^{2\pi}\int_0^{R} y^2 \, r\,dr\,d\theta \qquad y = r\sin\theta$$

$$I = \int_0^{2\pi}\int_0^{R} r^2 \sin^2\theta \,(r\,dr\,d\theta)$$

$$= \int_0^{R} r^3 dr \int_0^{2\pi} \sin^2\theta \, d\theta$$

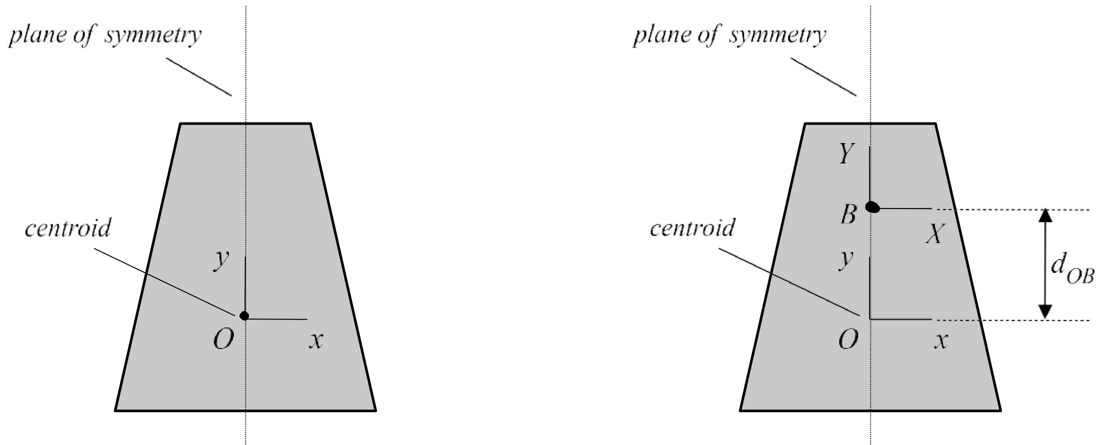
$$= \left[\frac{R^4}{4}\right]\left[\frac{1}{2}(2\pi)\right] = \frac{\pi}{4}R^4 = I$$

$I = \frac{\pi}{4}R^4$ bending

$I_p = \frac{\pi}{2}R^4$ torsion.

Second area moment of a cross section

Consider the beam cross section shown below left that is symmetrical about the y-axis but with no symmetry assumptions about the x-axis, where the origin of the x-y axis, O, is placed at the centroid of the cross section.



In the preceding derivation of the stress distribution across a cross section:

$$\sigma_x = -\frac{M\,y}{I_O} \tag{6}$$

we saw that this relationship depends on the "second area moment" I_O for the cross section:

$$I_O = \int_A y^2\, dA$$

where y is measured from the centroid of the cross section. Note that this parameter depends solely on the shape of the cross section and does not depend on either the material properties of the beam or the strain in the beam.

Tabulated expressions for the centroidal second area moments for a number of common beam cross sections are provided on the following pages.

For reasons that we will discuss later on, we often times need to know the second area moment about points on the plane of symmetry but not at the centroid of the cross section. Consider point B shown in the figure above right that is located at a distance d_{OA} from the centroid O on the plane of symmetry. Suppose we place a set of *X-Y* coordinate axes with its origin at A such that $X = x$ and $Y = y - d_{OB}$ Therefore, the second area moment about point B is found from:

$$I_B = \int_A y^2 dA = \int_A (y - d_{OB})^2 dA$$ $I_B = \int_A (y - d_{OB})^2 dA.$

$$= \int_A (y^2 - 2d_{OB}y + d_{OB}^2) dA$$

$$= \int_A y^2 dA - 2d_{OB}\int_A y\, dA + d_{OB}^2 \int_A dA$$

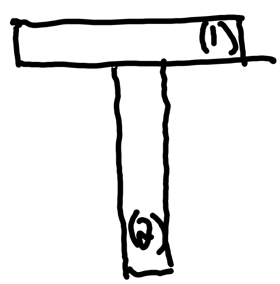
$$= I_O - 2d_{OB}\bar{y}A + Ad_{OB}^2$$

where A is the area of the cross section and $\bar{y}$ is the y-position of the centroid of the area. Since the origin O for the x-y axes is located at the centroid of the cross section, we have $\bar{y} = 0$. Therefore,

$$I_B = I_O + Ad_{OB}^2 \qquad (7)$$ $I_B = I_O + Ad^2_{OB}$

Equation (7) is the *"parallel axis theorem"* for second areas of moments. In words, in order to determine the second area moment about an arbitrary point B on the plane of symmetry, simply add Ad_{OB}^2 to the centroidal second area moment I_O, where d_{OB} is the distance between O and B.

In general, one needs to perform an integration over the cross section of the beam in order to evaluate this integral representation for I_O. We have seen this process in the earlier examples. However, for certain cross sections, we can use results from simple shapes to construct the overall second area moment for the cross section. To this end, we will need to use the above parallel axis theorem. This process is demonstrated in the following examples.



$I = I_1 + I_2.$

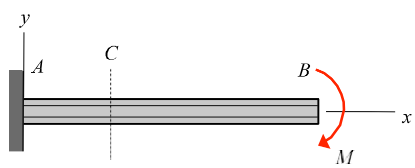
Example 10.5

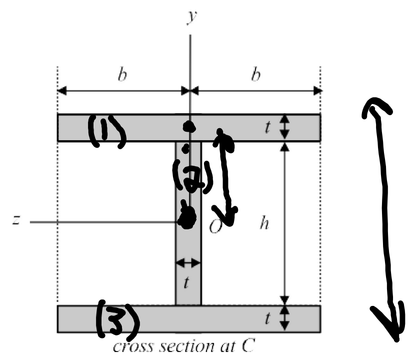
The cantilevered beam shown below is loaded in pure bending. The beam has a cross section at location C on the beam as shown below right. The origin O is located on the neutral axis of the beam.

a) Determine the second area moment I_{Oz} corresponding to the neutral axis of the beam.

b) Determine the distribution of normal stress on the cross section of the beam as a function of y.

c) Determine the maximum (magnitude) normal stress occurring on the cross-sectional face at C.





cross section at C

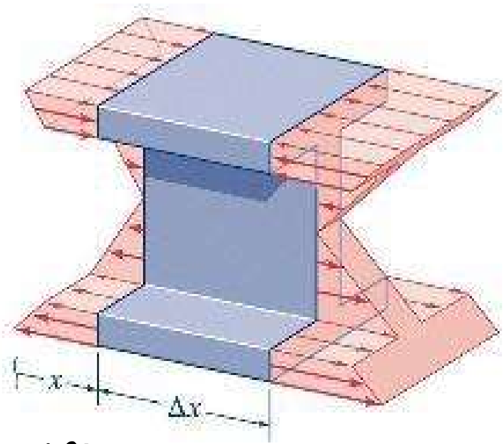
a) $\underline{I_O} = (I_1)_O + (I_2)_O + (I_3)_O$

$(I_2)_O = \frac{th^3}{12}$.

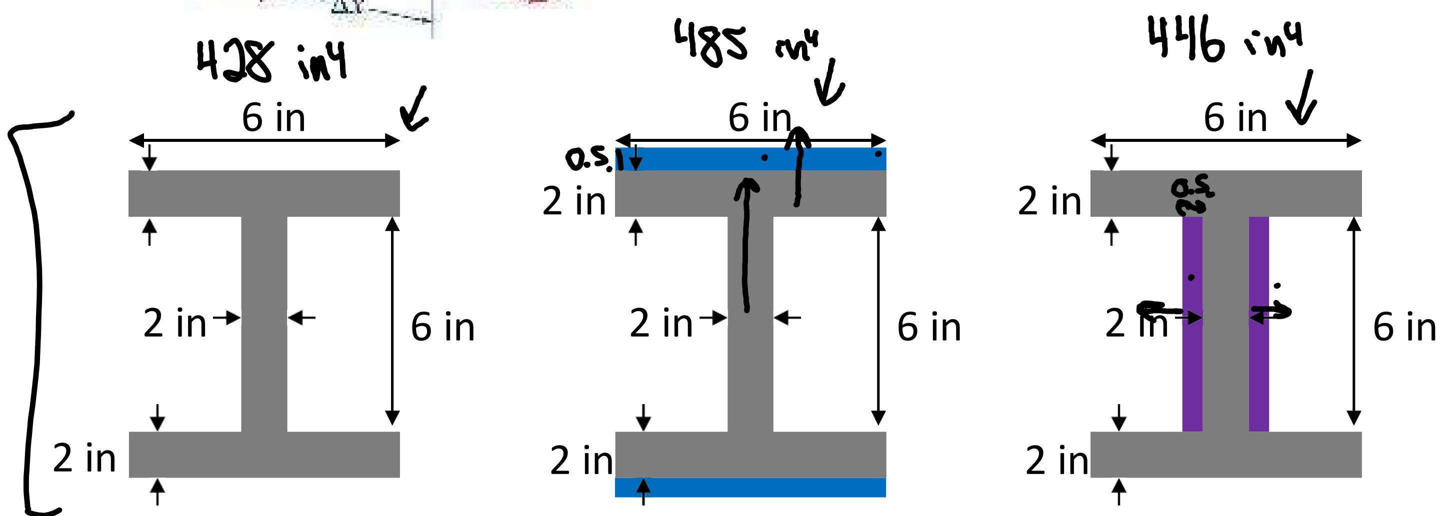
$(I_1)_O = I_1 + Ad_{OB}^2 = \frac{(2b)t^3}{12} + 2bt\left(\frac{h}{2} + \frac{t}{2}\right)^2$

$(I_3)_O = (I_1)_O$

Ben



How can we increase the strength of the beam?



$$I = \frac{2(6)^3}{12} + 2\left[\frac{6(2)^3}{12} + 6 * 2 * (4)^2\right]$$