16. Energy methods

Objectives:
To develop expressions for the strain energy for loaded structural elements and to use these expressions for the determination of elastic deformations in the structural elements due to the loadings.

Background:

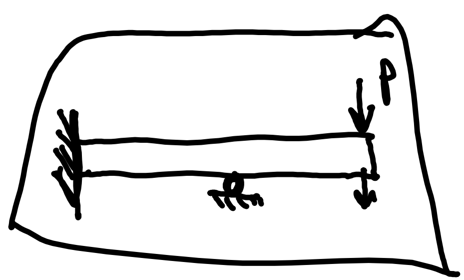
- *Work/energy equation*
  For a given system, the total work done on the system is equal to the change in total energy:
  $$W^{(nc)} = \Delta T + \Delta U$$
  where $W^{(nc)}$ is the work done on the system by non-conservative forces, ΔT is the change in kinetic energy and ΔU is the change in potential energy.

- *Equilibrium*
  For a system in equilibrium, the work/energy equation reduces to:
  $$W^{(nc)} = \Delta U$$
  which says that the change in potential energy is equal to the work done on the system.

- *Strain energy in springs*
  Recall that the potential energy in a spring is given by:
  $$U = \frac{1}{2}k\Delta^2$$
  where k is the stiffness of the spring and Δ is the stretch/compression in the spring. Since this potential energy results from the change in strain in the spring, this is often times referred to as the "strain energy" in the spring.

Lecture topics:

a) Expressions for strain energy in a structural element.
b) Using the work-energy principle for determining deflections.
c) Castigliano's second theorem for determinate structures.
d) Castigliano's second theorem for indeterminate structures.



Castigliano's.

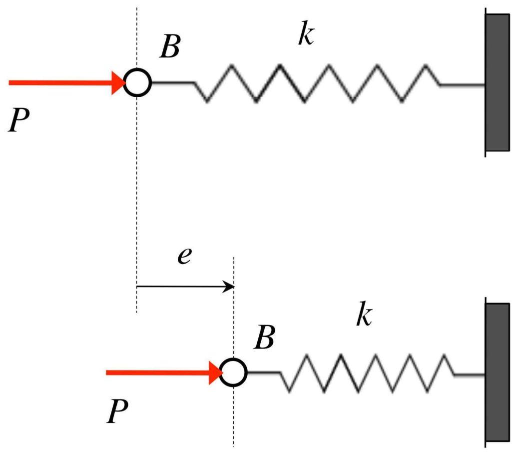
$$\Delta_i = \frac{\partial U}{\partial P_i} \qquad 0 = \frac{\partial U}{\partial R_i}$$

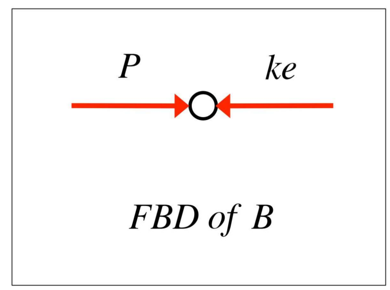
Lecture notes

a) Strain energy expressions

Motivating example #1

Consider a spring of stiffness k acted upon by a constant magnitude force P. Assume that the spring is uncompressed before the application of the force. Let e represent the compression in the spring resulting from the application of the force P . Write down the equilibrium form of the work energy equation for the system.





For equilibrium, we know that $P = ke$ for all deformations e. Therefore, the work done by P under equilibrium conditions is:

$$W^{(P)} = \int_0^e P\,de = k\int_0^e e\,de = \frac{1}{2}ke^2 = \frac{1}{2}Pe$$

We know that the potential (strain) energy in the spring can be written down directly as: $U = \frac{1}{2}ke^2$. However, for practice, let's derive this expression. To do so, recall that the change in potential of a conservative force is equal to the negative of the work done by the force:

$$W^{(sp)} = -\int_0^e ke\,de \quad ; \quad \text{"-"}\ \textit{since spring force opposes motion}$$

$$= -\frac{1}{2}ke^2$$

Therefore,

$$U = -W^{(sp)} = \frac{1}{2}ke^2 \quad \text{(which agrees with what we already knew above)}$$

From this, the work/energy equation for equilibrium is:

$$W^{(P)} = U \quad \Rightarrow \quad \frac{1}{2}Pe = \frac{1}{2}ke^2$$

$k = \frac{P}{e}$

Alternate representation:
Since we are considering equilibrium, $P = ke$, we could have written the strain energy in the spring as:

$$U = \frac{1}{2}ke^2 = \frac{1}{2}k\left(\frac{P}{k}\right)^2 = \frac{P^2}{2k}$$

This representation directly shows the dependence of the strain energy on both the applied load and the stiffness of the spring. For this expression, the work energy equation $W^{(P)} = U$ takes on the form of:

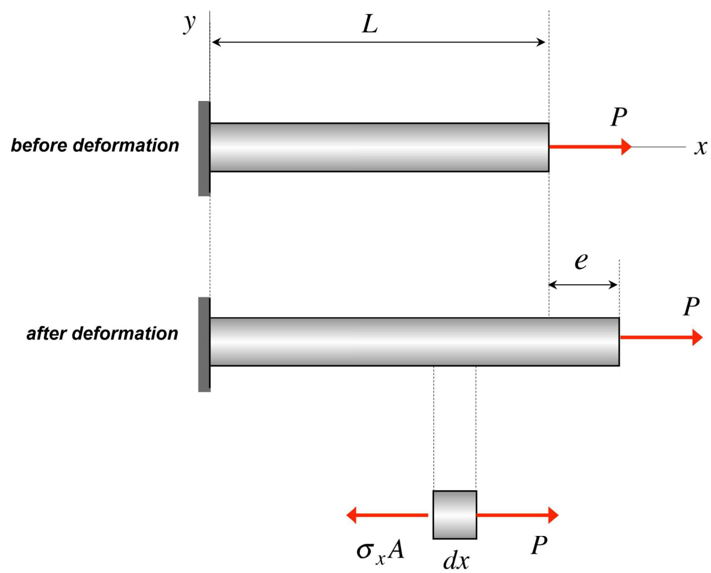
$$\frac{1}{2}Pe = \frac{P^2}{2k}$$

from which we can recover the expected expression for the static elongation of the spring:

$$e = \frac{P}{k}$$

Motivating example #2

Consider a straight rod (with constant cross-sectional area A) under the action of an extensive axial load P and fixed to ground on its left end. Determine an expression for the strain energy in the rod as a result of the axial load P.



The axial load P is related to the axial stress through:

$$P = \sigma_x A \tag{1}$$

For a linearly elastic material:

$$\sigma_x = E\varepsilon_x = E\frac{du}{dx} \tag{2}$$

And, the elongation in the rod is given by:

$$e = L\frac{du}{dx} \quad \Rightarrow \quad \frac{du}{dx} = \frac{e}{L} \tag{3}$$

Combining (1)-(3) gives:

$$P = \frac{EA}{L}e \qquad e = \frac{PL}{EA}. \tag{4}$$

The work done by the axial load P is given by:

$$W^{(P)} = \int_0^e P\,de = \int_0^e \frac{EA}{L}e\,de = \frac{1}{2}\frac{EA}{L}e^2 = \frac{1}{2}Pe \tag{5}$$

Since $U = W^{(P)}$, the strain energy in the rod is given by:

$$U = \frac{1}{2}Pe \tag{6}$$

Alternate representation:
From equation (1),

$$e = \frac{PL}{EA}$$

we can write the strain energy in the rod as:

$$U = \frac{1}{2}\frac{P^2 L}{EA} \qquad (7)$$

This representation directly shows the dependence of the strain energy on both the applied load and the material and properties of the rod. For this expression, the work energy equation $W^{(P)} = U$ takes on the form of:

$$\frac{1}{2}Pe = \frac{1}{2}\frac{P^2 L}{EA}$$

from which we can recover the expected expression for the static elongation of the spring:

$$e = \frac{PL}{EA}$$

General expressions for strain energy and work

The total strain energy for a linearly elastic body can be written as:

$$\underline{U} = \int_{vol} \bar{u}\, dV \qquad (8)$$

where:

$$\bar{u} = \textit{strain energy density function}$$

$$= \frac{1}{2}\left[\sigma_x\left(\varepsilon_x - \alpha\Delta T\right) + \sigma_y\left(\varepsilon_y - \alpha\Delta T\right) + \sigma_z\left(\varepsilon_z - \alpha\Delta T\right) + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right] \qquad (9)$$

From this general expression above, we will derive strain energy functions for a number of types of components, including: rods, shafts and bending beams.

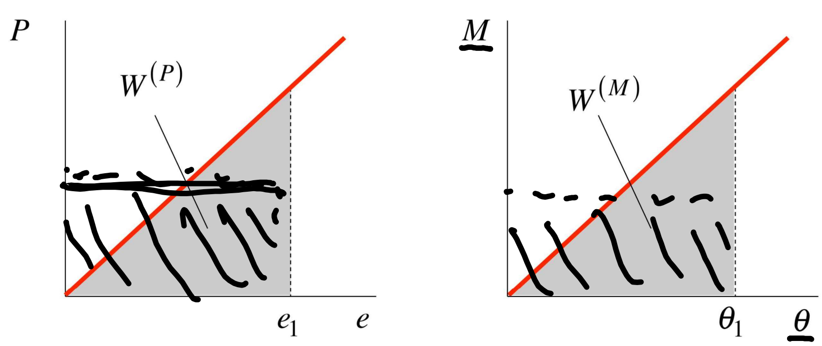
Also, recall that the work due a force P acting through a distance e_1 can be written as:

$$W^{(P)} = \int_0^{e_1} P\, de$$

And, the work due to a couple M acting through an angle θ_1 can be written as:

$$W^{(M)} = \int_0^{\theta_1} M\, d\theta$$

Suppose that these forces and moments act slowly (such that dynamic effects are not significant) and with *linear* relationships between P and e, and between M and θ, as indicated by the plots below.



In this case, the work due to P and and the work due to M (the areas under the respective P vs. e and M vs. θ curves) can be written as:

$$W^{(P)} = \frac{1}{2} P\left(e_1\right) e_1$$

$$\underline{W^{(M)}} = \frac{1}{2} M\left(e_1\right) \underline{\theta_1}$$

Note that the second expression above applies to both a torque T acting through a twist angle of ϕ and to a bending moment M acting through an angle of beam rotation θ.

Component: rod carrying axial load P

Here we consider a rod of length *L*, cross-sectional area *A* and Young's modulus *E* carrying an axial load of *P*. For axially-loaded rods, we have the following stress and strain functions:

$$\sigma_x = E(x)\frac{du(x)}{dx} \overset{OR}{=} \frac{P(x)}{A(x)}$$

$$\varepsilon_x = \frac{du(x)}{dx} \overset{OR}{=} \frac{P(x)}{A(x)E(x)}$$

and, in addition, $dV = A(x)dx$.

Substituting these into the general strain energy expression (8) gives *EITHER*:

$$U = \frac{1}{2}\int_0^L \sigma_x \varepsilon_x A\,dx = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx \tag{10a}$$

OR

$$U = \frac{1}{2}\int_0^L \sigma_x \varepsilon_x A\,dx = \frac{1}{2}\int_0^L \frac{P^2}{EA}dx \tag{10b}$$

where, in general, E , A , P and u may all be functions of x.

For the special case where E , A , P and u are all constants in x, expression (10b) reduces to:

$$\boxed{U = \frac{1}{2}\frac{P^2L}{EA}} \tag{10c}$$

as we derived earlier.

Component: circular shaft carrying torque T

Here we consider a circular cross section shaft of length *L*, polar area moment I_P and shear modulus *G* carrying a torque of *T*. For a circular shaft carrying a torque T along the x-axis, we have the following stress and strain functions:

$$\tau = G(x)\rho \frac{d\phi(x)}{dx} \overset{OR}{=} \frac{T(x)\rho}{I_P(x)}$$

$$\gamma = \rho \frac{d\phi(x)}{dx} \overset{OR}{=} \frac{T\rho}{GI_P}$$

where ρ is the radial distance from the centerline of the shaft cross section, ϕ is the angle of twist and, in addition, $dV = dA\,dx$.

Substituting these into the general strain energy expression (8) gives *EITHER*:

$$U = \frac{1}{2}\int_0^L \int_{area} \tau\gamma\, dA\, dx = \frac{1}{2}\int_0^L \left(\int_{area} \rho^2\, dA \right) G \left(\frac{d\phi}{dx} \right)^2 dx = \frac{1}{2}\int_0^L GI_P \left(\frac{d\phi}{dx} \right)^2 dx \qquad (11a)$$

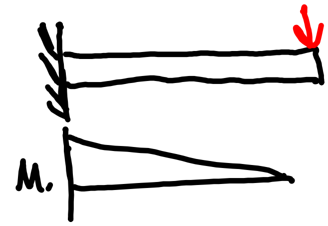
OR

$$U = \frac{1}{2}\int_0^L \int_{area} \tau\gamma\, dA\, dx = \frac{1}{2}\int_0^L \left(\int_{area} \rho^2\, dA \right) \frac{T^2}{GI_P{}^2} dx = \frac{1}{2}\int_0^L \frac{T^2}{GI_P} dx \qquad (11b)$$

where $I_P = \int_{area} \rho^2\, dA$. Here G, I_P, T and ϕ may all be functions of x.

For the special case where G, I_P, T and ϕ are all constants in x, expression (11b) reduces to:

$$\boxed{U = \frac{1}{2}\frac{T^2 L}{GI_P}} \qquad (11c)$$



Component: bending beam – flexural strain energy

Here we consider a thin beam of length L, second area moment I and Young's modulus modulus E. The transverse deflection of the beam is $v(x)$, the bending moment in the beam is $M(x)$ and with y being the cross sectional coordinate in the direction transverse to the beam. For a thin Euler-Bernoulli beam we have the following stress and strain components corresponding to the normal (flexural) stress:

$$\sigma_x = -E(x)y\frac{d^2v(x)}{dx^2} = -\frac{M(x)y}{I(x)}$$

$$\varepsilon_x = -y\frac{d^2v(x)}{dx^2} = \frac{M(x)y}{E(x)I(x)}$$

and, in addition, $dV = dA\,dx$. Here we will assume that the Young's modulus does not vary across the beam's cross-section. Substituting these into the general strain energy expression (8) gives *EITHER*:

$$U = \frac{1}{2}\int_0^L \int_{area} \sigma_x \varepsilon_x \, dA\,dx = \frac{1}{2}\int_0^L \left(\int_{area} y^2\,dA \right) E\left(\frac{d^2v}{dx^2}\right)^2 dx = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dx^2}\right)^2 dx \qquad (12a)$$

OR

$$U = \frac{1}{2}\int_0^L \int_{area} \sigma_x \varepsilon_x \, dA\,dx = \frac{1}{2}\int_0^L \left(\int_{area} y^2\,dA \right) \frac{M^2}{EI^2} dx = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx \qquad (12b)$$

where $I = \int_{area} y^2\,dA$. Here E, I, M and v may all be functions of x.

Component: bending beam – shear strain energy

For a bending beam, we also have energy that is attributed to shear stress/strain. For the same notation as before, we can write for the shear stress and shear strain:

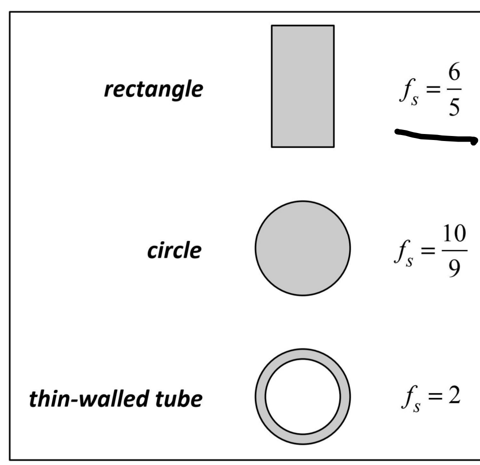
$$\tau_{xy} = \frac{V(x)Q(x,y)}{I(x)t(y)}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{1}{G}\frac{V(x)Q(x,y)}{I(x)t(y)}$$

Substituting into the general strain energy expression (8) gives:

$$U = \frac{1}{2}\int_0^L \int_{area} \tau_{xy}\gamma_{xy}\,dA\,dx$$

$$= \frac{1}{2}\int_0^L \left(\int_{area} \frac{Q^2(x,y)}{t^2(y)} dA \right) \frac{V^2}{GI^2} dx = \frac{1}{2}\int_0^L \frac{f_s V^2}{GA} dx$$



where:

$$f_s(x) = \frac{A(x)}{I^2(x)} \int_{area} \frac{Q^2(x,y)}{t^2(y)} dA = \text{"form factor" for the beam cross section}$$

Note that the form factor expression above has been calculated for some common cross-sections, as presented to the right.

Summary

The strain energy functions for the three types of members investigated here (axially-loaded members, torsionally-loaded members and members with flexural and shear stresses due to bending) are summarized below.

Member loading type	***Strain energy: load-based***	***Strain energy: displacement-based***
axial	$U = \frac{1}{2}\int_0^L \frac{F^2 dx}{EA} = \frac{1}{2}\frac{F^2 L}{EA}$	$U = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
torsion	$U = \frac{1}{2}\int_0^L \frac{T^2}{GI_p} dx = \frac{1}{2}\frac{T^2 L}{GI_p}$	$U = \frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
bending - flexural	$U_\sigma = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx$	$U_\sigma \frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
bending - shear	$U_\tau = \frac{1}{2}\int_0^L \frac{f_s V^2}{GA} dx$	

In this chapter, we will focus on the use of the load-based formulations of strain energy listed above. In a later chapter when we work with the finite element formulation, we will use the dispacement based formulation.

c) Work-energy equation

Recall that the work-energy equation for a system can be written as:

$$W = T + U$$

For static equilibrium, the change in kinetic energy T is zero. Therefore, the above reduces to:

$$W = U$$

The usage of the work-energy equation above is very limited in its usefulness in displacement analysis. For simple systems of having an applied load acting at only a single point, the work-energy equation can be used to determine the static deflection of the structure at the point at which the load is applied. For more complicated loads, we will still have only a single work-energy equation for loads at multiple points; however, we will need multiple equations to solve for displacements. In that case, we need to appeal to more advanced methods, such as Castigliano's methods that follow.

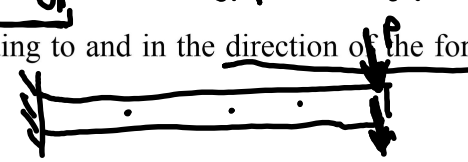
d) Castigliano's Second Theorem – applied to <u>determinate</u> structures

Consider a determinate linearly elastic deformable body or system acting upon by N forces P_i ; $i = 1,2,...,N$. Among all possible equilibrium configurations of the system, the actual configuration is the one for which:

$$\Delta_i = \frac{\partial U}{\partial P_i} \quad ; \quad i = 1,2,...,N$$

$$\delta = \frac{\partial U}{\partial P} \qquad \theta = \frac{\partial U}{\partial M} \qquad \phi = \frac{\partial U}{\partial T}$$

where Δ_i is the displacement corresponding to and in the direction of the force P_i, and U is the strain energy for the system.



e) Castigliano's Second Theorem – applied to <u>indeterminate</u> structures

Consider an indeterminate linearly elastic deformable body or system acting upon by N forces P_i ; $i = 1,2,...,N$. Since the system is indeterminate, there will be a number (N_R) of redundant forces in the strain energy function: R_i ; $i = 1,2,...,N_R$. Among all possible equilibrium configurations of the system, the actual configuration is the one for which:

$$\Delta_i = \frac{\partial U}{\partial P_i} \quad ; \quad i = 1,2,...,N$$

$$0 = \frac{\partial U}{\partial R_i} \quad ; \quad i = 1,2,...,N_R$$

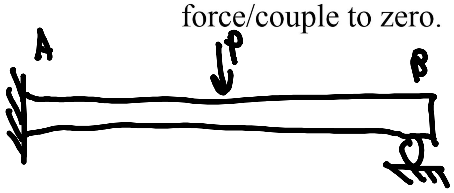
where Δ_i is the displacement corresponding to and in the direction of the force P_i (or R_i), and U is the strain energy for the system.

Comments on the usage of Castigliano's theorem for deflection analysis

a) For *determinate structures*, one is able to solve for the external reactions directly through equilibrium equations. As a result, it is possible to also find the internal resultants (such as shear forces, axial forces and bending moments), and consequently, the strain energy for the structure can be written in terms of only the applied forces that are used for find deflections.

b) For *indeterminate structures*, one has too few equilibrium equations for determining the external reactions on the structure; therefore, it is not possible to find internal resultants, and, consequently, the strain energy will include many of these unknown reactions. Suppose that the structure of interest has an indeterminacy of order N_R; that is, there are N_R too few equations available for finding reactions. Therefore, we have N_R redundant forces/couples. For these problems, one needs to first choose which reactions that will be considered redundant, and write the equilibrium equations so that the remaining reactions are in terms of these redundant forces/couples. The additional N_R equations needed for determining the reactions are found from the second Castigliano equation above: $\boxed{0 = \partial U / \partial R_i}$; $i = 1,2,...,N_R$. Once these reactions are found, then the first set of Castigliano equations are used to find the desired deflections.

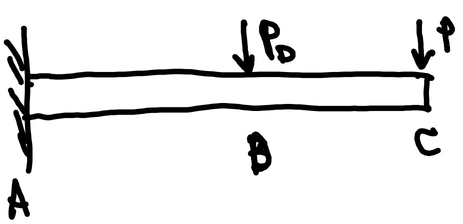
c) Note that Castigliano's theorem allows us to determine components of displacements only at points where loadings are applied and only components of displacements that are aligned with the loadings. If the structure is not acted upon by a force at a point and/or along a line of action for which deflections are needed, we simply need to apply a "dummy" force/couple to the structure, treating as a regular applied load. After applying Castigliano's theorem, then set the dummy force/couple to zero.



$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx + \frac{1}{2}\int_0^L \frac{f_s V^2}{GA}\,dx.$$

$M = f(P, L, B_y)$ $\quad$ $V = f(P, L, B_y)$.

$\frac{\partial U}{\partial B_y} = 0 \Rightarrow$ another eqn for B_y.



$$\frac{\partial U}{\partial P} = \delta_C$$

$$\left[\frac{\partial U}{\partial P_D}\right]_{P_D = 0} = \delta_B.$$

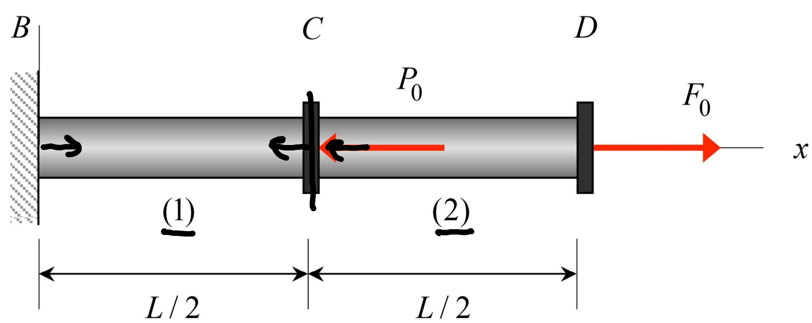
Example 16.1

A rod having a solid cross section of area A and made up of a material with a Young's modulus of E is made up of components (1) and (2). Components (1) and (2) are joined by rigid connector C, with component (1) being attached to rigid wall at end B and with a second connector at end D of (2). Loads P_0 and F_0 act on connectors C and D.

a) Determine the strain energy stored in the rod in terms of the applied loads and the work done by the applied loads under static equilibrium conditions.

b) Write down the work-energy equation for the system under static equilibrium conditions. Explain why the *work-energy method* cannot be used directly to determine the static displacements of either C or D.

c) Use *Castigliano's theorem* to determine the static displacements of C and D.



$$\left(\Sigma F_x\right)_D = F_0 - F_2 = 0 \Rightarrow F_2 = F_0$$

$$\left(\Sigma F_x\right)_C = F_2 - F_1 - P_0 = 0 \Rightarrow F_1 = F_0 - P_0$$

$$U_{tot} = U_1 + U_2.$$

$$= \frac{1}{2}\frac{F_1^2 (L/2)}{EA} + \frac{1}{2}\frac{F_2^2 (L/2)}{EA}$$

$$= \frac{1}{2}\frac{(F_0 - P_0)^2 L}{2EA} + \frac{1}{2}\frac{F_0^2 L}{2EA}.$$

$$u_D = \frac{\partial U}{\partial F_0} = \frac{1}{2}\frac{(F_0 - P)L}{EA}(1) + \frac{1}{2}\frac{F_0 L}{EA}.$$ in positive x.

$$u_C = \frac{\partial U}{\partial P_0} = \frac{1}{2}\frac{(F_0 - P_0)L}{EA}(-1)$$ in neg x.

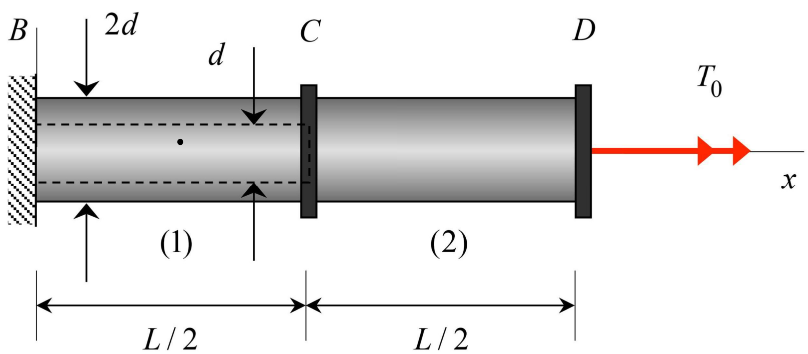
Example 16.2

A shaft (made up of a material with a Young's modulus of E) is composed of elements (1) and (2), where (1) is a hollow circular tube and (2) has a solid circular cross section. Elements (1) and (2) are joined by a rigid connector C, with (1) attached to fixed wall at B and (2) joined to a rigid connector at D. A torque T_0 is applied to connector D.

a) Determine the strain energy stored in the shaft in terms of the applied torque T_0 and the work done by the applied torque under static equilibrium conditions.

b) Write down the work-energy equation for the system under static equilibrium conditions. Use the *work-energy method* to determine the static rotation of connector D.

c) Use *Castigliano's theorem* to determine the static rotation of D.



a) $U = U_1 + U_2$

$$U = \frac{1}{2}\frac{T_1^2(L/2)}{GI_{p1}} + \frac{1}{2}\frac{T_2^2(L/2)}{GI_{p2}}$$

$$(\Sigma T)_D = T_0 - T_2 = 0 \Rightarrow T_2 = T_0$$

$$(\Sigma T)_C = T_2 - T_1 = 0 \Rightarrow T_1 = T_0$$

$$I_{p1} = \frac{\pi}{2}\left[\left(\frac{2d}{2}\right)^4 - \left(\frac{d}{2}\right)^4\right] = \frac{15}{32}\pi d^4$$

$$I_{p2} = \frac{\pi}{2}\left[\frac{2d}{2}\right]^4 = \frac{\pi}{2}d^4$$

c) $$\phi_D = \frac{\partial U}{\partial T_0} = \frac{1}{2}\frac{T_0 L}{GI_{p1}} + \frac{1}{2}\frac{T_0 L}{GI_{p2}}$$