Summary

The strain energy functions for the three types of members investigated here (axially-loaded members, torsionally-loaded members and members with flexural and shear stresses due to bending) are summarized below.

Member loading type	***Strain energy: load-based***	***Strain energy: displacement-based***
axial	$U=\frac{1}{2}\int_0^L \frac{F^2 dx}{EA}$	$U=\frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx$
torsion	$U=\frac{1}{2}\int_0^L \frac{T^2}{GI_p}dx$	$U=\frac{1}{2}\int_0^L GI_p\left(\frac{d\phi}{dx}\right)^2 dx$
bending - flexural	$U_\sigma=\frac{1}{2}\int_0^L \frac{M^2}{EI}dx$	$U_\sigma \frac{1}{2}\int_0^L EI\left(\frac{d^2u}{dx^2}\right)^2 dx$
bending - shear	$U_\tau=\frac{1}{2}\int_0^L \frac{f_s V^2}{GA}dx$	

In this chapter, we will focus on the use of the load-based formulations of strain energy listed above. In a later chapter when we work with the finite element formulation, we will use the dispacement based formulation.

$$\boxed{\Delta_i = \frac{\partial U}{\partial P_i} \qquad 0 = \frac{\partial U}{\partial R_i}}$$

Deflection analysis – Castigliano's method

The procedure for deflection analysis using Castigliano's method:

i) First determine if you need to include any "dummy" loads (recall that the Castigliano's method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). *Add in ALL of the needed dummy loads from the start*; this can save you a lot of time down the road.

ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.

- If *DETERMINATE*, solve these equations for the external reactions.
- If *INDETERMINATE*, establish the "order" N_R of the indeterminancy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of n redundant reactions (R_i ; $i = 1,2,...,N_R$). *Write the remaining reactions in terms of these N_R redundant reactions.*

iii) Divide beam into sections: $x_i < x < x_{i+1}$. This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.). V, M, F, T.

iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M_i(x)$, shear force $V_i(x)$ and axial force $F_{Ni}(x)$ through that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_i = \frac{1}{2EI}\int_{x_i}^{x_{i+1}} M_i^2\,dx + \frac{f_s}{2GA}\int_{x_i}^{x_{i+1}} V_i^2\,dx + \frac{1}{2EA}\int_{x_i}^{x_{i+1}} F_{Ni}^2\,dx$$

From these strain energy terms, write down the total strain energy for the structure: $U = U_1 + U_2 + U_3 + \ldots$. It is recommended that you do NOT expand out the "squared" terms in these integrals at this point.

v) If the problem is *INDETERMINATE*, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

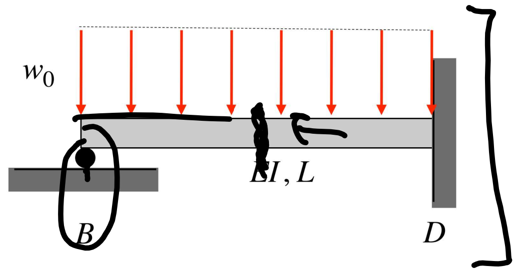
$$\boxed{0 = \frac{\partial U}{\partial R_i}} \; ; \quad i = 1,2,...,N_R$$

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above. ⇒ solve for reactions.

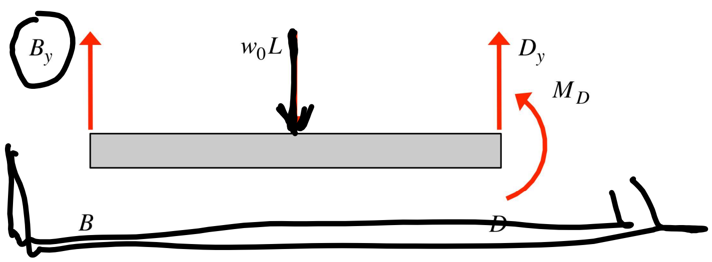
vi) Determine the desired deflections/rotations using Castigliano's method: $\delta_i = \partial U / \partial P_i$. Be sure to set any dummy loads to zero in the end.

Example 16.7

Determine the reaction at end B of the beam shown.



Equilibrium – FBD of entire beam



From here, we see that the problem is statically indeterminate: 3 unknowns (B_y, D_y and M_D) and only two equations. Here, we will choose B_y to be our redundant reaction:

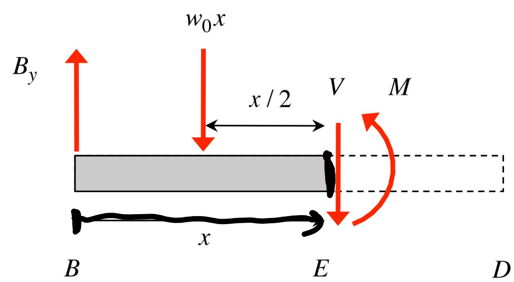
$$\sum F_y = B_y - w_0 L + D_y = 0 \;\; \Rightarrow \;\; D_y = w_0 L - B_y$$

$$\sum M_B = -(w_0 L)\left(\frac{L}{2}\right) + D_y L + M_D = 0 \;\; \Rightarrow \;\; M_D = -\left(w_0 L - B_y\right)L + \frac{1}{4} w_0 L^2$$

3 unknowns

2 eqns

Determining internal bending moment



$$\sum M_E = M - B_y x + (w_0 x)\left(\frac{x}{2}\right) = 0 \;\; \Rightarrow \;\; M(x) = -\frac{w_0 x^2}{2} + B_y x$$

Strain energy in beam (ignoring contributions from shear stress/strain)

$$U = \frac{1}{2}\int_0^L \frac{M^2}{EI} dx = \frac{1}{2EI}\int_0^L \left(-\frac{w_0 x^2}{2} + B_y x\right)^2 dx$$

$$\frac{\partial U_M}{\partial B_y} = \frac{1}{EI}\int M \frac{\partial M}{\partial B_y} dx.$$

Castigliano's theorem

With B_y being our choice for the redundant reaction:

$$\boxed{0 = \frac{\partial U}{\partial B_y}} = \frac{1}{2EI}\int_0^L 2\left(-\frac{w_0 x^2}{2} + B_y x\right)(x)\,dx = \frac{1}{EI}\int_0^L \left(-\frac{w_0 x^3}{2} + B_y x^2\right)dx$$

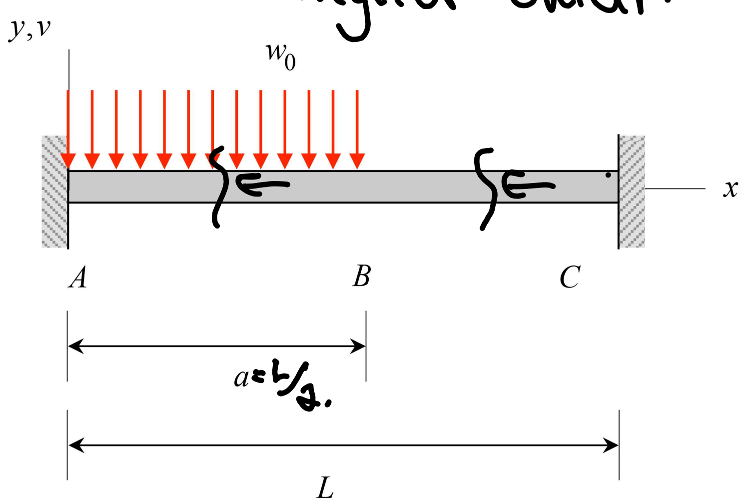
$$= \frac{1}{EI}\left[-\frac{w_0 x^4}{8} + \frac{B_y x^3}{3}\right]_{x=0}^{x=L} = \frac{1}{EI}\left[-\frac{w_0 L^4}{8} + \frac{B_y L^3}{3}\right] \Rightarrow \underbrace{B_y = \frac{3}{8} w_0 L}$$

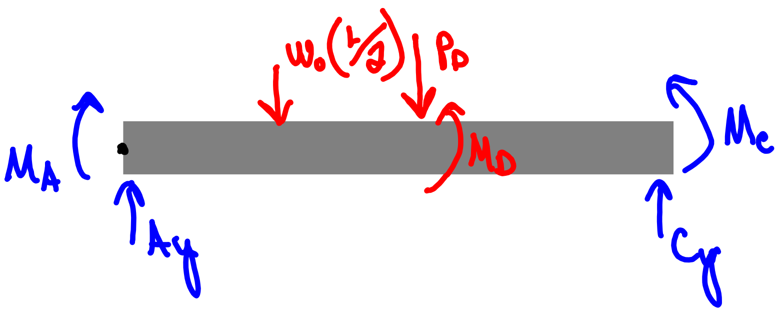
Problem B

Find the vertical deflection of the beam at point B and the angle of rotation of the beam at B. Let E and I be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.

Neglect shear.

$$\frac{\partial U}{\partial P} = \delta \qquad \frac{\partial U}{\partial M} = \theta$$





$$(1)\quad (\Sigma M)_A = -M_A + M_C + M_D - w_0\left(\tfrac{L}{2}\right)\left(\tfrac{L}{4}\right) - P_D\left(\tfrac{L}{2}\right) + C_y L = 0$$

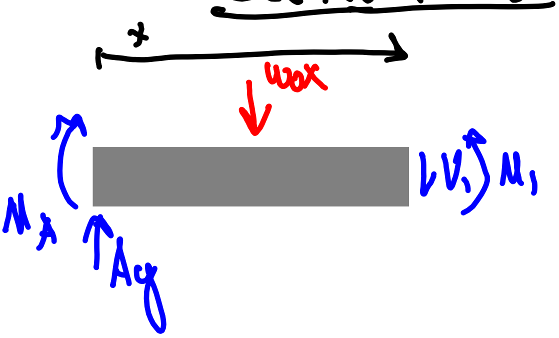
$$(2)\quad \Sigma F_y = A_y + C_y - w_0\left(\tfrac{L}{2}\right) - P_D = 0$$

4 unknowns

2 eqns.

2

Section AB

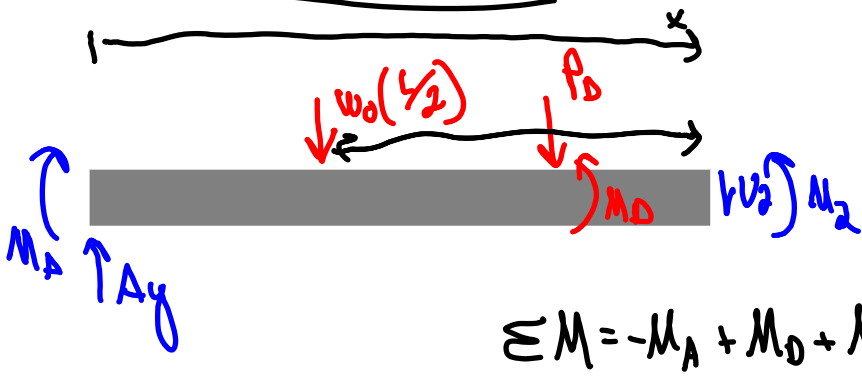


$$\Sigma F_y = A_y - w_0 x - V_1 = 0$$

$$\Sigma M = -M_A + M_1 - A_y x + w_0 x\left(\tfrac{x}{2}\right) = 0$$

$$M_1(x) = M_A + A_y x - w_0 \frac{x^2}{2}$$

Section BC



$$\Sigma M = -M_A + M_D + M_2 - A_y x + w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right) + P_D\left(x - \frac{L}{2}\right) = 0$$

$$M_2(x) = M_A - M_D + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right) - P_D\left(x - \frac{L}{2}\right)$$

$$U = U_1 + U_2$$

$$U = \frac{1}{2EI}\int_0^{L/2} M_1^2\,dx + \frac{1}{2EI}\int_{L/2}^{L} M_2^2\,dx$$

Find reactions

$$\left[\frac{\partial U}{\partial A_y}\right]_{P_D, M_D = 0} = 0 = \frac{1}{EI}\int_0^{L/2} M_1 \frac{\partial M_1}{\partial A_y}\,dx + \frac{1}{EI}\int_{L/2}^{L} M_2 \frac{\partial M_2}{\partial A_y}\,dx.$$

$$\frac{\partial M_1}{\partial A_y} = x \qquad \frac{\partial M_2}{\partial A_y} = x$$

$$0 = \frac{1}{EI}\int_0^{L/2}\left(M_A + A_y x - w_0\frac{x^2}{2}\right)x\,dx + \frac{1}{EI}\int_{L/2}^{L}\left[M_A + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right)\right]x\,dx. \quad (3)$$

$$\left[\frac{\partial U}{\partial M_A}\right]_{P_D, M_D = 0} = 0 \qquad \frac{\partial M_1}{\partial M_A} = 1 \qquad \frac{\partial M_2}{\partial M_A} = 1.$$

$$0 = \frac{1}{EI}\int_0^{L/2}\left(M_A + A_y x - w_0\frac{x^2}{2}\right)dx + \frac{1}{EI}\int_{L/2}^{L}\left[M_A + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right)\right]dx. \quad (4)$$

Eqns (1-4): solve for reactions

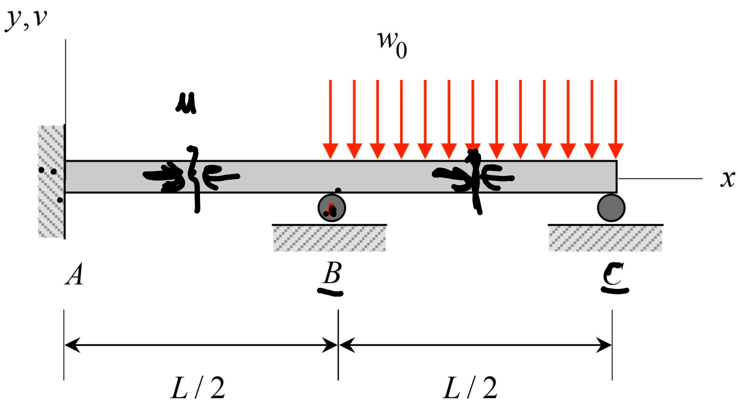
Find the displacements:

$$\frac{\partial M_1}{\partial P_D} = 0 \qquad \frac{\partial M_2}{\partial P_D} = -\left(x - \frac{L}{2}\right)$$

$$\delta_B = \left[\frac{\partial U}{\partial P_D}\right]_{P_D, M_D = 0} = 0 + \frac{1}{EI}\int_{L/2}^{L}\left[M_A + A_y x - w_0\left(\frac{L}{2}\right)\left(x - \frac{L}{4}\right)\right]\left[-\left(x - \frac{L}{2}\right)\right]dx.$$

Problem C

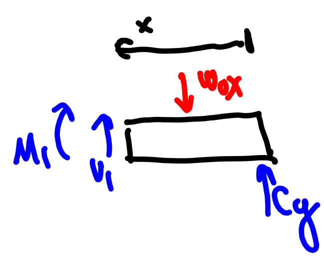
Determine the reactions at rollers B and C on the beam below. Let E and I be the Young's modulus and second area moment of the beam cross section, respectively, of the beam.



$$(\Sigma M)_A = -M_A + B_y\left(\tfrac{L}{2}\right) + C_y L - w_0\left(\tfrac{L}{2}\right)\left(\tfrac{3L}{4}\right) = 0$$

$$(\Sigma F_y) = A_y + B_y + C_y - w_0\left(\tfrac{L}{2}\right) = 0$$

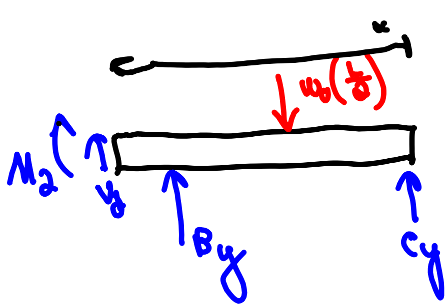
4 unknowns, 2 eqns.



$$(\Sigma M)_1 = -M_1 + C_y x - w_0\frac{x^2}{2} = 0$$

$$M_1(x) = C_y x - w_0\frac{x^2}{2}$$

$$\Sigma F_y = V_1 + C_y - w_0 x = 0 \Rightarrow V_1(x) = -C_y + w_0 x.$$



$$(\Sigma M)_2 = -M_2 + B_y\left(x - \tfrac{L}{2}\right) + C_y x - w_0\left(\tfrac{L}{2}\right)\left(x - \tfrac{L}{4}\right) = 0$$

$$\underline{M_2(x)} = B_y\left(x - \tfrac{L}{2}\right) + C_y x - w_0\left(\tfrac{L}{2}\right)\left(x - \tfrac{L}{4}\right)$$

$$\Sigma F_y = V_2 + B_y + C_y - w_0\left(\tfrac{L}{2}\right) = 0$$

$$V_2(x) = -B_y - C_y + w_0\left(\tfrac{L}{2}\right)$$

$$U = \frac{1}{2EI}\int_0^{L/2} M_1^2\,dx + \frac{f_s}{2GA}\int_0^{L/2} V_1^2\,dx + \frac{1}{2EI}\int_{L/2}^{L} M_2^2\,dx + \frac{f_s}{2GA}\int_{L/2}^{L} V_2^2\,dx.$$

Neglect shear.

$$\frac{\partial M_1^2}{\partial B_y} = (2M_1)\frac{\partial M_1}{\partial B_y}$$

$$\frac{\partial U}{\partial B_y} = 0 = \frac{1}{EI}\int_0^{L/2} M_1\frac{\partial M_1}{\partial B_y}\,dx + \frac{1}{EI}\int_{L/2}^{L} M_2\frac{\partial M_2}{\partial B_y}\,dx.$$

$$\frac{\partial M_1}{\partial B_y} = 0 \qquad \frac{\partial M_2}{\partial B_y} = (x - L/2)$$

$$0 = \frac{1}{EI}\int_{L/2}^{L}\left[B_y(x - L/2) + C_y x - w_0\left(\frac{L}{2}\right)(x - L/4)\right](x - L/2)\,dx.$$

$$\frac{\partial U}{\partial C_y} = 0$$

Example 16.8

For the following examples, ***set up*** the problem for determining the requested deflections using Castigliano's method:

- draw appropriate FBDs;
- determine internal results for each section;
- set up the integrals for calculating the required deflections;
- explain how Castigliano's method is used to solve. Discuss the application of dummy forces (when needed) and how to handle redundant forces for indeterminate structures.

Problem A

Find the load carried by member (2) of the structure below. Let E and A be the Young's modulus and cross-sectional area, respectively, of member (2), whereas E and I are the Young's modulus and second area moment of the cross section of (1), respectively.