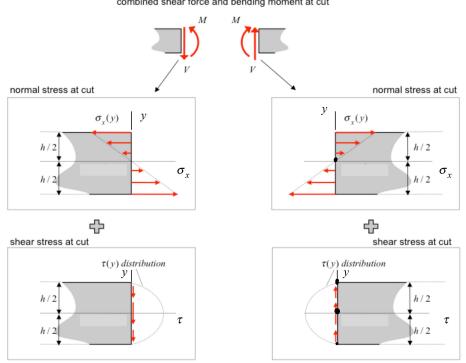
Summary: stress distribution due to combined shear force and bending couple at cut

At a cut through a section of a beam experiencing both a shear force V and bending moment M, and abiding by the Euler-Bernoulli assumptions, we can make the following observations (see following figure):

- a) Both normal stresses σ_r and shear stresses τ exist at the cut.
- b) The normal stresses vary linearly in the *y*-direction as in the pure bending case. All previous observations about the normal stresses due to pure bending also apply in the case.
- c) The shear stresses are approximately constant in the *z*-direction (into the depth of the beam) for "narrow beams", t > 2h.
- d) The shear stress is zero at the outer surfaces of the beam.
- e) For rectangular cross-section beams, the shear stress distribution at a cut is parabolic in the y-direction:

$$\tau = \frac{6}{Ah^2} \left(\frac{h^2}{4} - y^2 \right) V$$

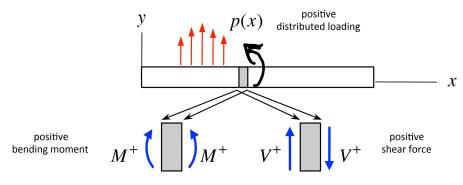
where A is the area of the cross section. The maximum shear stress, $\tau_{max} = 3V / 2A$, occurs at the neutral axis (y = 0).



combined shear force and bending moment at cut

Graphical method for constructing shear force and bending moment diagrams

Sign conventions:



Basic relationships (as derived via equilibrium relations):

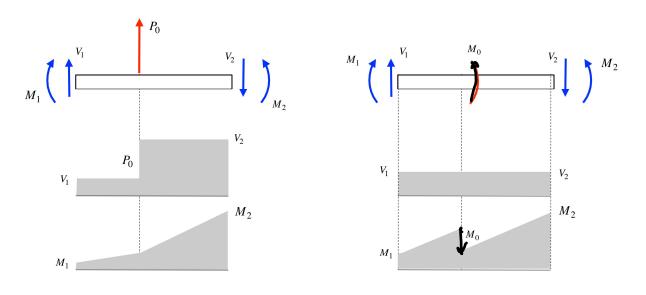
$$\frac{dV}{dx} = p(x) \qquad \Rightarrow \qquad V_2 = V_1 + \int_{x_1}^{x_2} p(x) dx$$
$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M_2 = M_1 + \int_{x_1}^{x_2} V(x) dx$$

Concentrated shear force V_0 applied at location x:

 $V(x^+) = V(x^-) + V_0$ (jump UP in shear force)

Concentrated moment M_0 applied at location x:

 $M(x^+) = M(x^-) - M_0$ (jump DOWN in moment)



Deflections of statically-determinate beams – DEFINITE INTEGRAL APPROACH Recall that for statically-determinate beams, we can determine the external reactions on the beam using the rigid body equilibrium equations. Assume that for a given determinate problem we have already determined these external reactions through equilibrium analysis. Using these, our goal is to determine the deflection of the beam over the full length of the beam.

To this end, we will now reconsider equations (1), (2), (5) and (7) above. We will integrate these equations over a given segment $x_1 < x < x_2$ of the beam. Note that the following results assume that the cross sectional and material properties are constant throughout a given segment.

Equation (1):

$$\frac{dV}{dx} = \underline{p(x)} \qquad \Rightarrow \qquad V(x) = V(x_1) + \int_{x_1}^x p(s) ds \tag{10}$$

Equation (2):

$$\frac{dM}{dx} = V(x) \qquad \Rightarrow \qquad M(x) = M(x_1) + \int_{x_1}^x V(s) ds \tag{11}$$

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Equation (7):

$$EI\frac{d\theta}{dx} = M(x) \implies \theta(x) = \theta(x_1) + \frac{1}{EI}\int_{x_1}^x M(s)ds$$
(12)

Equation (5):

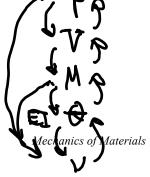
$$\frac{dv}{dx} = \theta(x) \qquad \Rightarrow \qquad v(x) = v(x_1) + \int_{x_1}^x \underbrace{\theta(s)}_{x_1} ds \qquad (13)$$

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These results can be used two alternate ways for determining the deflection of a beam:

- i) <u>Fourth-order approach</u> Here we start with the loading p(x) and perform the four integrations of (10)-(15) to obtain v(x).
- ii) <u>Second-order approach</u> Here we determine bending moment distribution M(x) through FBDs and equilibrium analysis. With this M(x), equations (12)-(13) are used to produce the deflection v(x).

Note that with this *definite integral* approach, the boundary conditions such as $\theta(x_1)$ and $v(x_1)$ naturally appear in the solutions.



Summary: beam deflection – second-order integration method

The procedure to determine the deflection of a bending beam using the second-order integration method:

- i) Before starting, write down the boundary conditions (BCs) for the problem.
- ii) Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions. If *DETERMINATE*, solve these equations for the external reactions.
- iii) Divide beam into sections: $x_i < x < x_{i+1}$, where this section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).
- iv) For each section, draw free an FBD of either the left of right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment M(x) through that section of the beam.
- v) Use the following integrals to determine slope and deflection of the beam over $x_i < x < x_{i+1}$:

$$\theta(x) = \theta(x_i) + \frac{1}{EI} \int_{x_i}^x M(x) dx$$
$$v(x) = v(x_i) + \int_{x_i}^x \theta(x) dx$$

- vi) The final values of slope and displacement for one section $\theta(x_{i+1})$ and $v(x_{i+1})$ become the initial values of slope and displacement for the next section (continuity conditions).
- vii) Enforce any remaining boundary conditions to determine any remaining integrations constants. For the case of *INDETERMINATE* beams, additional equations needed for determining external reactions are also produced through the enforcement of boundary conditions. These equations are solved with the equilibrium equations in ii) above.

Deflection analysis – Castigliano's method

The procedure for deflection analysis using Castigliano's method:

- → i) First determine if you need to include any "dummy" loads (recall that the Castigliano's method can produce deflections/rotations at points on the structures at which applied forces/moments act and in directions in which these forces/moments act). Add in ALL of the needed dummy loads from the start; this can save you a lot of time down the road.
- Draw a free body diagram (FBD) of the entire structure, and from this FBD write down the equilibrium equations; these equilibrium equations will be written in terms of the external reactions.
 - If *DETERMINATE*, solve these equations for the external reactions.
 - If *INDETERMINATE*, establish the "order" N_R of the indeterminancy (i.e., equal to the number of additional equations needed to solve for external reactions). From your external reactions, choose a set of *n* redundant reactions (R_i ; $i = 1, 2, ..., N_R$). Write the remaining reactions in terms of these N_R redundant reactions.
 - iii) Divide beam into sections: $x_i < x < x_{i+1}$. This section division is dictated by: support reactions, beam geometry changes, and/or load changes (concentrated forces/moments, line load definition changes, etc.).
 - iv) For each section, draw an FBD of either the left or right side of the body from a cut through that section of the beam. From this FBD, determine the distribution of bending moment $M_i(x)$, shear force $V_i(x)$ and axial force $F_{Ni}(x)$ through that section of the structure. Using these, write down the strain energy in that section of the structure using:

$$U_{i} = \frac{1}{2EI} \int_{x_{i}}^{x_{i+1}} M_{i}^{2} dx + \frac{f_{s}}{2GA} \int_{x_{i}}^{x_{i+1}} V_{i}^{2} dx + \frac{1}{2EA} \int_{x_{i}}^{x_{i+1}} F_{Ni}^{2} dx$$

From these strain energy terms, write down the total strain energy for the structure: $U = U_1 + U_2 + U_3 + \dots$ It is recommended that you do NOT expand out the "squared" terms in these integrals at this point.

v) If the problem is *INDETERMINATE*, first set up the additional algebraic equations for the reactions of the problems using Castigliano:

$$0 = \frac{\partial U}{\partial R_i} \quad ; \quad i = 1, 2, \dots, N_R$$

Be sure to set any dummy loads to zero in the end. Solve these equations with the equilibrium equations from i) above.

vi) Determine the desired deflections/rotations using Castigliano's method: $\delta_i = \partial U / \partial P_i$. Be sure to set any dummy loads to zero in the end.



Summary

The strain energy functions for the three types of members investigated here (axiallyloaded members, torsionally-loaded members and members with flexural and shear stresses due to bending) are summarized below.

Member loading type	Strain energy: load-based	Strain energy: displacement-based
axial	$U = \frac{1}{2} \int_{0}^{L} \frac{F^2 dx}{EA}$	$U = \frac{1}{2} \int_{0}^{L} EA\left(\frac{du}{dx}\right)^{2} dx$
torsion	$U = \frac{1}{2} \int_{0}^{L} \frac{T^2}{GI_p} dx$	$U = \frac{1}{2} \int_{0}^{L} GI_{p} \left(\frac{d\phi}{dx}\right)^{2} dx$
bending - flexural	$U_{\sigma} = \frac{1}{2} \int_{0}^{L} \frac{M^2}{EI} dx$	$U_{\sigma} \frac{1}{2} \int_{0}^{L} EI\left(\frac{d^{2}u}{dx^{2}}\right)^{2} dx$
bending - shear	$U_{\tau} = \frac{1}{2} \int_{0}^{L} \frac{f_s V^2}{GA} dx$	

In this chapter, we will focus on the use of the load-based formulations of strain energy listed above. In a later chapter when we work with the finite element formulation, we will use the dispacement based formulation.

<u>Method</u>

Consider the following steps in setting up and solving the finite element displacement equations:

- Defining the nodes and elements for the problem. Choose a set of N+1 nodes along the length of the rod at locations $x_1(=0), x_2, x_3, ..., x_N, x_{N+1}(=L)$. The subdomain of $x_i < x < x_{i+1}$ is known as the ith element of length $L_i = x_{i+1} - x_i$, for i = 1, 2, ..., N. The value of $k_i = (EA)_i / L_i$ is determined through the average value of EA over the ith element and the element length L_i .
- <u>Constructing the global stiffness matrix</u>. Construct the stiffness matrix [K]. The resulting matrix will be tri-diagonal and of size $(N+1) \times (N+1)$.
- <u>Constructing the force vector</u> Construct the force vector $\{F\}$ as being made up on the resultant *external* force acting on each node. The resulting vector will be of length N+1.
- <u>Enforcing fixed-displacement boundary conditions</u>. The fixed-displacement boundary conditions are enforced through the elimination of appropriate terms in the resulting stiffness matrix [K] and forcing vector $\{F\}$. For example, if the ith node has a fixed (zero) displacement, we eliminate the ith row and ith column of [K] and the ith row of $\{F\}$. If the problem has "n" fixed nodal displacements, then the stiffness matrix and force vector will be of sizes $(N n + 1) \times (N n + 1)$ and N n + 1, respectively¹.
- <u>Solving</u>. The nodal displacements u_k ; k = 1, 2, ..., N n + 1 are found from the solution of the algebraic equilibrium equations:

$$\begin{bmatrix} K \end{bmatrix} \{ u \} = \{ F \}$$

through a linear equation solver in an application such as Matlab or Mathematica.

• <u>Stress calculations</u>

The average stress across the ith element is found from:

$$\sigma_i = \frac{E_i}{L_i} \left(u_{i+1} - u_i \right)$$

A Matlab code for constructing the stiffness matrix and force vector for a general N-element finite element mesh, for enforcing fixed-displacement boundary conditions and solving for the displacement of the non-fixed nodes is shown on the following page.



¹ If no displacement boundary conditions are applied, the rod will be physically unconstrained against motion. Consequently, there will be no equilibrium solution possible. The stiffness matrix for an unconstrained system will be singular, and therefore, non invertible.

ME 323 – Mechanics of Materials

Exam 2

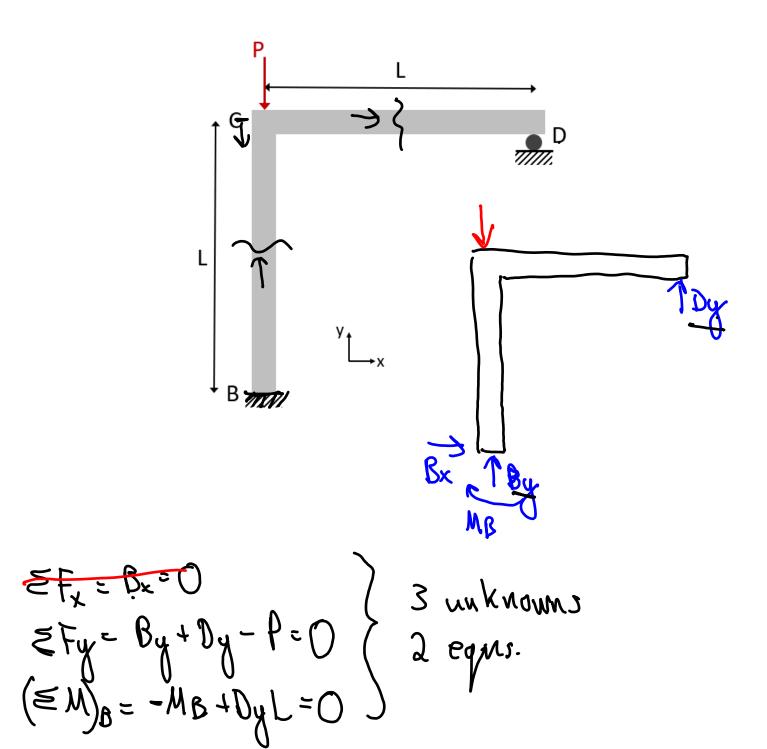
March 29, 2023

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PROBLEM #3 (25 points)

An L-shaped beam is rigidly fixed at B and supported by a roller at D. A point load is applied at C in the y-direction. You can assume that the value of P is small enough that BC does not buckle. You can neglect shear energy in your analysis.

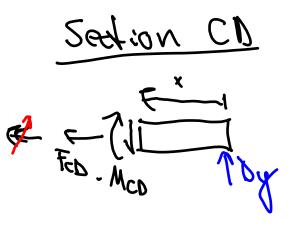
- (a) Make a free body diagram and write the equilibrium equations.
- (b) Use Castigliano's Second Theorem to determine the reactions at B and D.
- (c) Use Castigliano's Second Theorem to determine the displacement of point C in the y-direction.



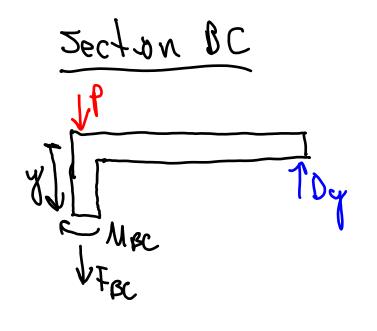
March 29, 2023

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 $\sum_{w \in D} \frac{W_{CD}(x)}{W} = x D y^{-1}$ $EF_x = 0 = -F_{co}$

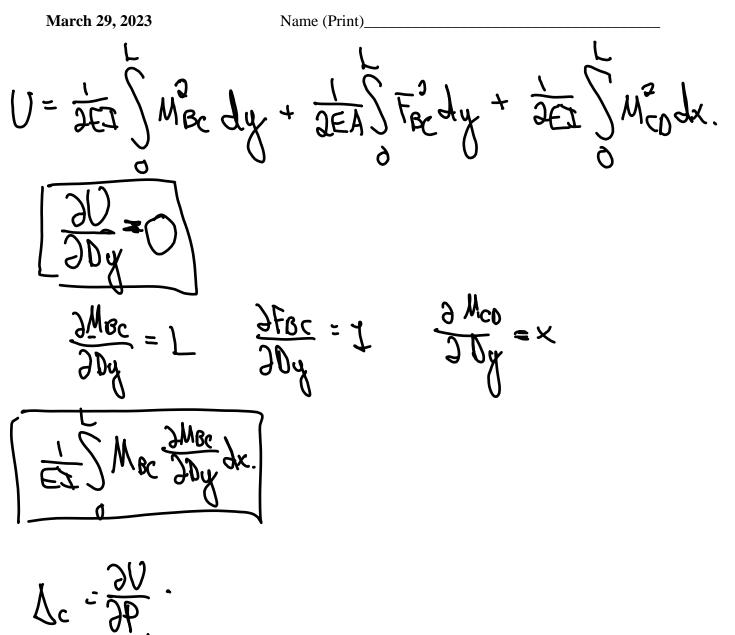


$$EM = 0 = -MBC + LDy$$

$$M_{BC}(x) = LDy$$

$$EFy = 0 = -F_{BC} - P + Dy$$

$$F_{BC} = Dy - P$$



ME 323 – Mechanics of Materials

Exam 2

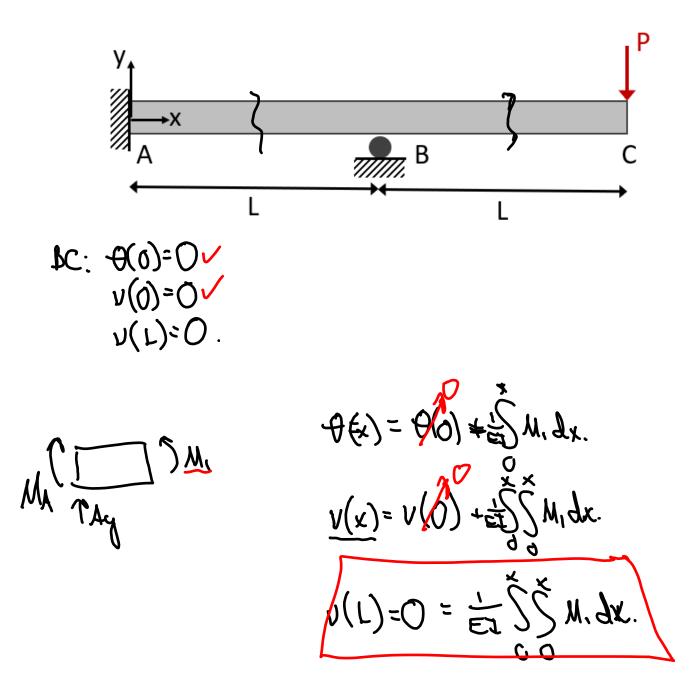
March 29, 2023

Name (Print)_____

PROBLEM #1 (25 points)

Beam ABC is fixed at end A and is supported by a roller support at B. A concentrated force P acts at C. E and I are constant along the beam. **Use the second order integration method** to calculate the following:

- (a) Draw a free body diagram and write the equilibrium equations.
- (b) Find the reactions on the beam at A and B in terms of P.
- (c) Find the equation for the vertical displacement, v(x) using the x-direction shown in the figure, throughout the beam in terms of P, L, E, and I.
- (d) Find the slope (θ) at point B in terms of P, L E, and I.



March 29, 2023

Name (Print)_____

March 29, 2023

Name (Print)_____

ME 323 – Mechanics of Materials

Exam 2

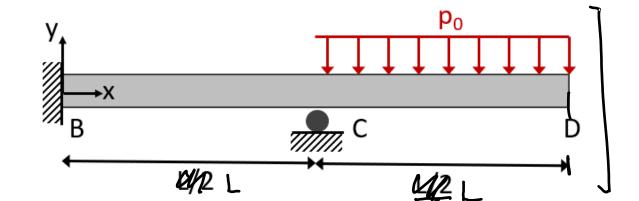
March 29, 2023

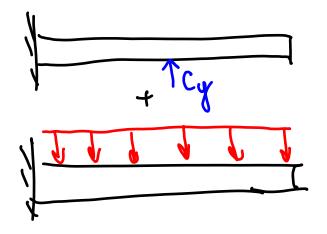
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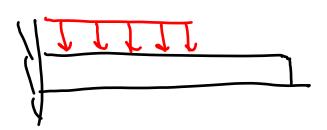
PROBLEM #2 (25 points)

A cantilever beam BCD has a distributed load p_0 acting between C and D.

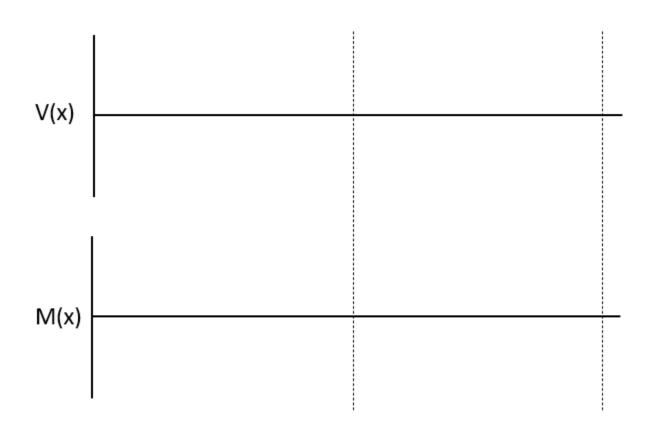
- (a) Draw a free body diagram and write the equilibrium equations.
- (b) Use the superposition principle and the superposition tables provided to calculate the values of the reactions at B and C. Leave your answers in terms of p_0 and L.
- (c) Draw the internal moment M(x) and shear force V(x) along the beam on the axes on the next page. Label the values of M(x) and V(x) at points B, C, and D.







March 29, 2023	Name (Print)
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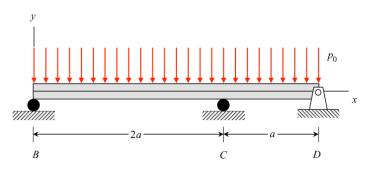
March 29, 2023

Name (Print)_____

April 11, 2018

PROBLEM NO. 4 - PART A – 7 points max.

A beam is made up a material with a Young's modulus of E and has a constant cross section with a second area moment of I. A downward, constant line load p_0 (force/length) acts along the full length of the beam. The beam has roller supports at B and C, along with a pin joint support at end D. Using the *superposition* approach, determine the reaction force acting on the beam at the roller support C.



SOLUTION

ME 323 Examination #2	Name
November 14, 2017	Instructor

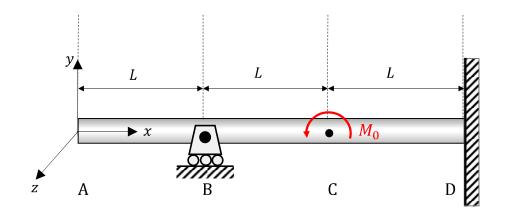
Page 2 of 15

PROBLEM NO. 1 – 30 points max.

The cantilever beam AD of the bending stiffness EI is subjected to a concentrated moment M_0 at C. The beam is also supported by a roller at B. Using Castigliano's theorem:

- a) Determine the reaction force at the roller B.
- b) Determine the rotation angle of the beam about z axis at the end A.

Ignore the shear energy due to bending. Express your answers in terms of M_0 , E, and I.

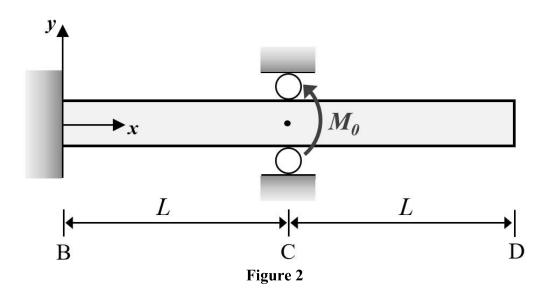


ME 323 Examination # 2	Name		
	(Print)	(Last)	(First)

PROBLEM #2 (25 points)

The beam BCD is fixed to the wall at B and supported by a roller at C. An external moment M_{θ} is applied at C. The beam has Young's modulus *E* and second moment of area *I*.

- a) Draw a free-body diagram of the entire beam, and write down the equilibrium equations.
- b) Use the second-order (or fourth-order) integration method to find the slope v'(x) and deflection v(x) of each segment of the beam. These can be left in terms of the unknown support reactions.
- c) Write down the relevant boundary conditions and continuity conditions for the beam.
- d) Use the boundary/continuity conditions to determine the reactions at B and C in terms of M_{θ} and L.
- e) Determine the deflection at the free end D. Sketch the deflection curve over the length of the beam. The sketch does not need to be exact; show enough detail to indicate the boundary conditions.



ME 323 Examination # 2	Name		
	(Print)	(Last)	(First)

PROBLEM #4 (25 Points):

PART A – 4 points

Figure 4A shows a beam that is subjected to point load at multiple locations. The beam has a T-shaped cross section as shown in Figure 4B. Circle the correct answer for the following questions:

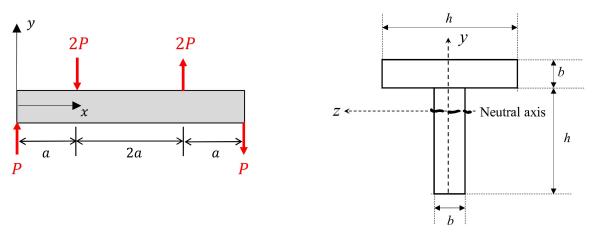
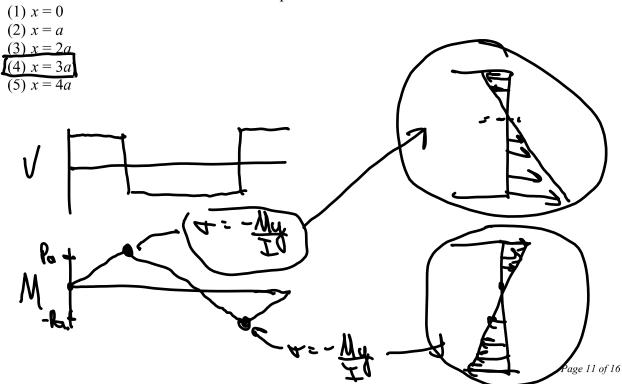


Figure 4A

Figure 4B

- a) On which cross section the maximum tensile stress is attained?
 - (1) x = 0(2) x = a (3) x = 2a (4) x = 3a
 - (5) x = 4a
- b) On which cross section the maximum compressive stress is attained?





School of Mechanical Engineering

ME 323 – Mechanics of Materials Final Exam December 16, 2023

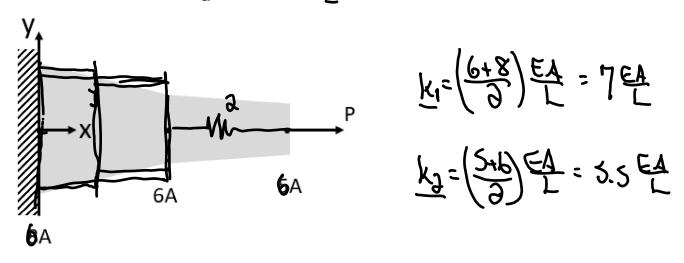
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Part D (5 points):

An axial assembly will be approximated using a finite element model using two elements. Both elements have a length of \underline{L} and a modulus of \underline{E} .



(a) (**3 points**) Using a finite element approximation with 3 nodes, fill out the stiffness matrix below:

$$[K] = \begin{bmatrix} \frac{k_1}{K_1} & \frac{-k_1}{K_1} & \frac{O}{K_2} \\ \frac{-k_1}{O} & \frac{-k_2}{K_2} & \frac{-k_2}{K_2} \end{bmatrix} = \begin{bmatrix} 4 & 7 \\ -K & 5.5 \end{bmatrix}$$

(b) (**2 points**) If the number of nodes increases from 3 to 12, how will that affect the error compare to the analytical simulation:

```
error_{3nodes} < error_{12nodes}
```

 $error_{3nodes} = error_{12nodes}$

 $error_{3nodes} > error_{12nodes}$

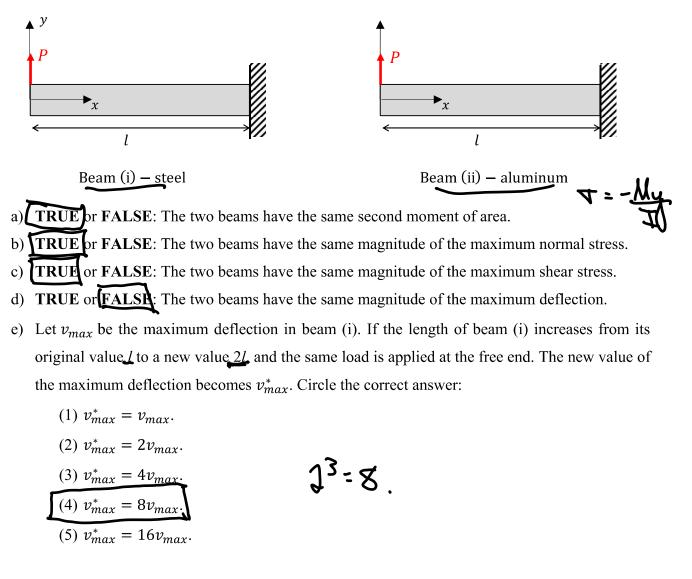
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ME 323 Examination # 2	Name			
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PROBLEM #4 (cont.):

PART B – 9 points

Beam (i) and (ii) shown below are identical, except that beam (i) is made of steel, and beam (ii) is made of aluminum. Note that $E_{\text{steel}} > E_{\text{aluminum}}$.



ME 323 – Mechanics of Materials

Exam 2

March 29, 2023

Name (Print)_____

PROBLEM #4 – PART A (6 points)

Beam (i) and (ii) are identical cylindrical beams except that beam (i) is made of steel and beam (ii) is made of aluminum. $E_{steel} > E_{aluminum}$.



Beam (a) Steel

(a) Circle the correct relationship between the maximum shear stresses in the two beams (**1 point**).

$$\begin{aligned} |\tau_{max,a}| < |\tau_{max,b}| \\ |\tau_{max,a}| &= |\tau_{max,b}| = 0 \\ |\tau_{max,a}| &= |\tau_{max,b}| \neq 0 \\ |\tau_{max,a}| > |\tau_{max,b}| \end{aligned}$$

(c) Circle the correct relationship between the maximum deflection v(x) in the two beams (1 point).

$$|v_{max,a}| < |v_{max,b}|$$
$$|v_{max,a}| = |v_{max,b}| = 0$$
$$|v_{max,a}| = |v_{max,b}| \neq 0$$
$$|v_{max,a}| > |v_{max,b}|$$

Beam (b) Aluminum

(b) Circle the correct relationship between the maximum shear stresses in the two beams (**1 point**).

$$\begin{aligned} |\sigma_{max,a}| &< |\sigma_{max,b}| \\ |\sigma_{max,a}| &= |\sigma_{max,b}| = 0 \\ |\sigma_{max,a}| &= |\sigma_{max,b}| \neq 0 \\ |\sigma_{max,a}| &> |\sigma_{max,b}| \end{aligned}$$

(d) The diameter of the original beams is D. If the diameter is doubled to 2D, how will the new deflection of the new beam (v_{max}^*) with diameter of 2D compare to the deflection of the original beam (v_{max}) with diameter of D (**3 points**):

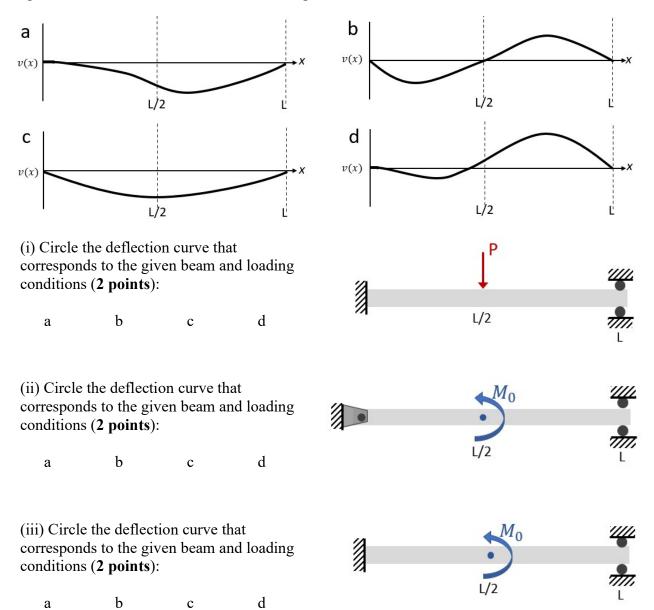
$$v_{max}^* = v_{max}$$
$$v_{max}^* = 2v_{max}$$
$$v_{max}^* = 4v_{max}$$
$$v_{max}^* = 8v_{max}$$
$$v_{max}^* = 16v_{max}$$

March 29, 2023

Name (Print)_

PROBLEM 4 – PART B (6 points)

Figures a-d indicate the deflection curve along four different beams.

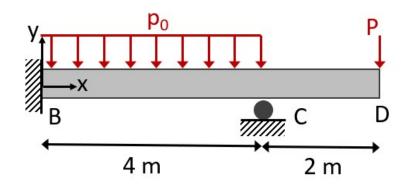


March 29, 2023

Name (Print)_

PROBLEM 4 – PART D (6 points)

A beam is loaded with a distributed load from 0 to 4 m and a point load at 6 m.



Circle the value(s) that will be zero at x = 0m (**2 points**):

V(0) M(0) $\theta(0)$ v(0)

Circle the value(s) that will be zero at x = 4m (2 points):

V(4) M(4) $\theta(4)$ v(4)

Circle the value(s) that will be zero at x = 6m (**2 points**):

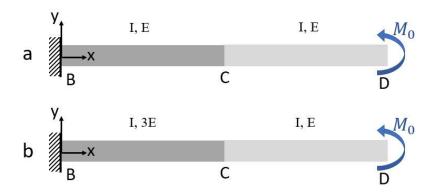
V(6) M(6) $\theta(6)$ v(6)

March 29, 2023

Name (Print)_

PROBLEM 4 – PART E (5 points)

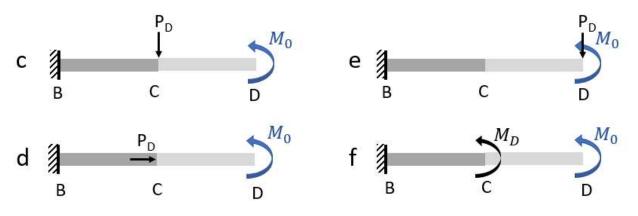
A simple cantilever is composed of two sections with an applied moment at the end.

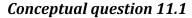


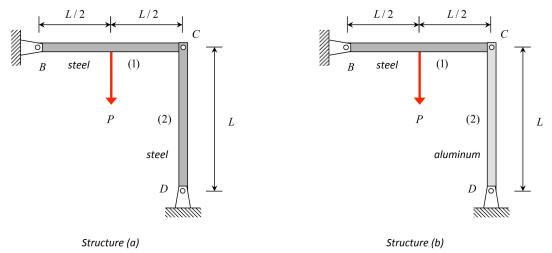
(i) (3 points) In beam (a), the two sections both have the same Young's moduli of E. In beam (b), one of the sections has a Young's modulus of 3E, while one has a Young's modulus of E. How does the total strain energy of these two beams compare?:

 $U_{total,a} > U_{total,b}$ $U_{total,a} = U_{total,b}$ $U_{total,a} < U_{total,b}$

(ii) (2 points) Circle the loading condition below (c to f) that would be used if we want to calculate the deflection at point C in the y-direction.







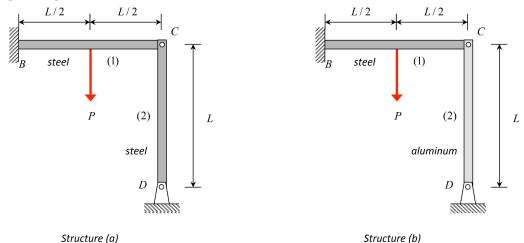
Structures (a) and (b) are identical, except that member (2) in Structure (a) is made of steel and member (2) in Structure (b) is made of aluminum. Let $(\sigma_2)_a$ and $(\sigma_2)_b$ represent the axial stresses in member (2) of Structures (a) and (b), respectively, due to the load P acting on member (1). Circle the item below that describes the relative sizes of these stresses:

i) $|(\sigma_2)_a| = |(\sigma_2)_b|$

ii)
$$|(\sigma_2)_a| \neq |(\sigma_2)_b|$$

iii) more information about the structures is needed in order to answer this question

Conceptual question 11.2



Structures (a) and (b) are identical, except that member (2) in Structure (a) is made of steel and member (2) in Structure (b) is made of aluminum. Let $(\sigma_2)_a$ and $(\sigma_2)_b$ represent the axial stresses in member (2) of Structures (a) and (b), respectively, due to the load P acting on member (a). Circle the item below that describes the relative sizes of these stresses:

- i) $|(\sigma_2)_a| = |(\sigma_2)_b|$
- ii) $|(\sigma_2)_a| \neq |(\sigma_2)_b|$
- iii) more information about the structures is needed in order to answer this question