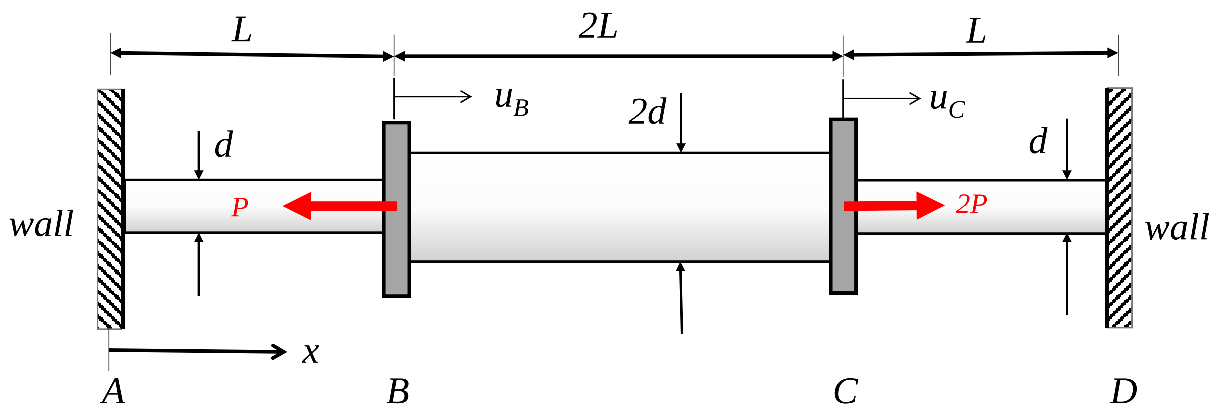
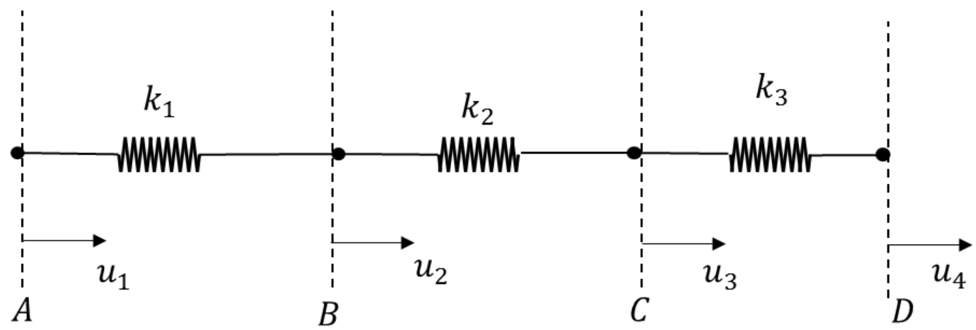
Q1 (10 Points): Consider the rod below with the Young's modulus *E*. Using a three-element finite element model,

a. Construct the global stiffness matrix.
b. Construct the force vector.
c. Enforce the displacement boundary condition.
d. Solve the nodal displacements.
e. Calculate the internal forces for each segment.



Solution



$$k_1 = \frac{E(A_1)}{L_1} = \frac{\pi d^2 E}{4L}; \qquad k_2 = \frac{E(A_2)}{L_2} = \frac{\pi d^2 E}{2L}; \qquad k_3 = \frac{E(A_3)}{L_3} = \frac{\pi d^2 E}{4L};$$

$$[K] = \begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1 + k_2 & -k_2 & 0 \\ 0 & -k_2 & k_2 + k_3 & -k_3 \\ 0 & 0 & -k_3 & k_3 \end{bmatrix} = \frac{\pi d^2 E}{L} \begin{bmatrix} 0.25 & -0.25 & 0 & 0 \\ -0.25 & 0.75 & -0.5 & 0 \\ 0 & -0.5 & 0.75 & -0.25 \\ 0 & 0 & -0.25 & 0.25 \end{bmatrix}$$

$$\{F\} = \begin{Bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{Bmatrix}$$

Since $u_1 = u_4 = 0$ (fixed wall), the corresponding rows and columns of the stiffness matrix and the 1st and 4th entries in the force column vector can be struck off.

$$\therefore [K] = \frac{\pi d^2 E}{L}\begin{bmatrix} 0.75 & -0.5 \\ -0.5 & 0.75 \end{bmatrix} \qquad \{F\} = \begin{Bmatrix} -1 \\ 2 \end{Bmatrix} P$$

Nodal displacements

$$\{u\} = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = [K]^{-1}\{F\} = \frac{PL}{0.3125\pi d^2 E}\begin{bmatrix} 0.75 & 0.5 \\ 0.5 & 0.75 \end{bmatrix}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix} = \frac{PL}{0.3125\pi d^2 E}\begin{Bmatrix} 0.25 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{\pi d^2 E}\begin{Bmatrix} 0.8 \\ 3.2 \end{Bmatrix}$$

Forces in members

$$F_1 = k_1(u_2 - u_1) = \frac{\pi d^2 E}{4L} \times \frac{PL}{\pi d^2 E}(0.8 - 0) = 0.2P$$

$$F_2 = k_2(u_3 - u_2) = \frac{\pi d^2 E}{2L} \times \frac{PL}{\pi d^2 E}(3.2 - 0.8) = 1.2P$$

$$F_3 = k_2(u_4 - u_3) = \frac{\pi d^2 E}{4L} \times \frac{PL}{\pi d^2 E}(0 - 3.2) = -0.8P \quad (comp)$$