ME 323: Mechanics of Materials **Homework 1**

Spring 2025 **Due: January 24, 2025**

Problem 1 (10 points):

The propped cantilever ABC shown in the figure is subjected to a distributed load w_0 over half of its length. The beam of constant E,I is supported by an elastic rod BD.

a) Use the Castigliano's theorem to determine the **deflection of point B** on the beam.
b) Plot M(x) and V(x) across the beam AC. Mark the critical values in the diagrams such as the maximum and minimum values, and locations of their zero values.

Ignore the shear energy due to bending in your analysis. Express answers in terms of given parameters E,I,L,A,w_0

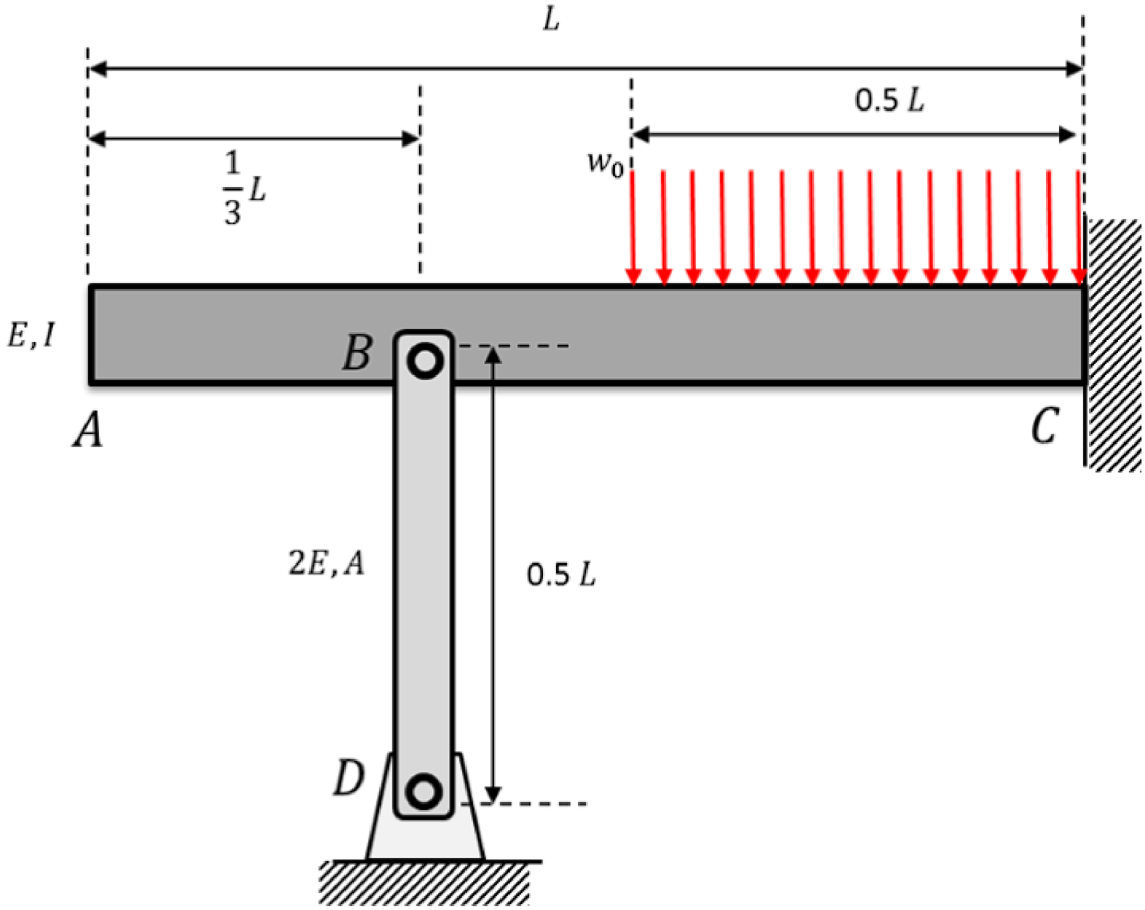
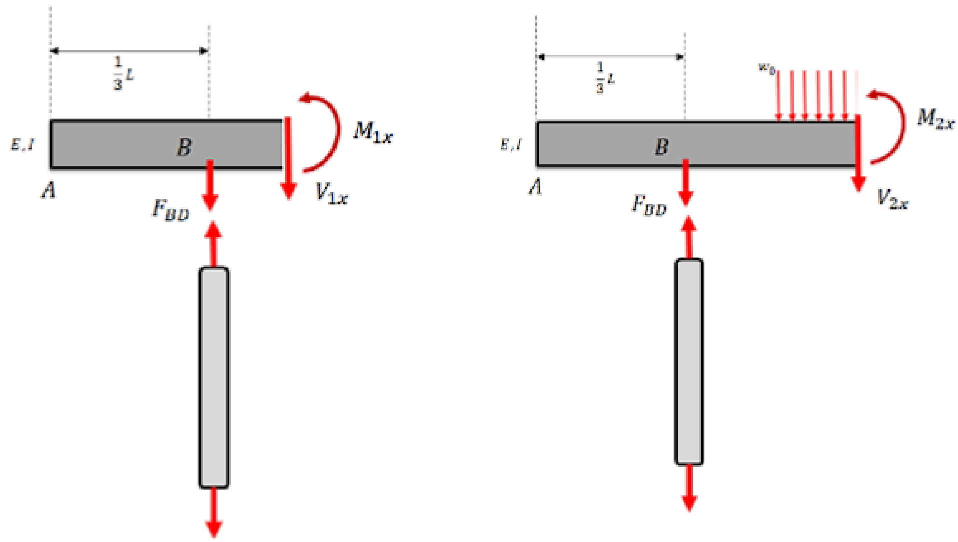


Figure 1: Beam for Problem 1 (Fall 17)



FBD analysis,

$\frac{1}{3}L < x < 0.5L$:

$$\Sigma F_y = -V_{1x} - F_{BD} = 0 \Rightarrow V_{1x} = -F_{BD}$$

$$\Sigma M_o = M_{1x} + F_{BD}\left(x - \frac{L}{3}\right) = 0 \Rightarrow M_{1x} = -F_{BD}\left(x - \frac{L}{3}\right)$$

$0.5L < x < L$:

$$\Sigma F_y = -V_{2x} - w_0(x - \frac{L}{2}) - F_{BD} = 0 \Rightarrow V_{2x} = -w_0(x - \frac{L}{2}) - F_{BD}$$

$$\Sigma M_o = M_{2x} + \frac{w_0}{2}(x - L/2)^2 + F_{BD}\left(x - \frac{L}{3}\right) = 0 \quad \Rightarrow M_{2x} = -\frac{w_0}{2}\left(x - \frac{L}{2}\right)^2 - F_{BD}\left(x - \frac{L}{3}\right)$$

$$U_{total} = \int_{L/3}^{0.5L} \frac{{M_{1x}}^2 dx}{2EI} + \int_{0.5L}^{L} \frac{{M_{2x}}^2 dx}{2EI}$$

$$= \frac{1}{2EI}\int_{L/3}^{0.5L} \left(F_{BD}\left(x-\frac{L}{3}\right)\right)^2 dx + \frac{1}{2EI}\int_{0.5L}^{L} \left(\frac{w_0}{2}\left(x-\frac{L}{2}\right)^2 + F_{BD}\left(x-\frac{L}{3}\right)\right)^2 dx$$

$$v_B = \frac{\partial U_{total}}{\partial(-F_{BD})} = \frac{1}{2EI}\int_{L/3}^{0.5L} \frac{\partial\left(F_{BD}\left(x-\frac{L}{3}\right)\right)^2}{\partial(-F_{BD})} dx + \frac{1}{2EI}\int_{0.5L}^{L} \frac{\partial\left(\frac{w_0}{2}\left(x-\frac{L}{2}\right)^2 + F_{BD}\left(x-\frac{L}{3}\right)\right)^2}{\partial(-F_{BD})} dx$$

F_{BD} pointing is -ve y

$$= -\left[\frac{1}{2EI}\int_{L/3}^{0.5L} 2F_{BD}\left(x-\frac{L}{3}\right)^2 dx + \frac{1}{2EI}\int_{0.5L}^{L} 2\left(\frac{w_0}{2}\left(x-\frac{L}{2}\right)^2 + F_{BD}\left(x-\frac{L}{3}\right)\right)\left(x-\frac{L}{3}\right) dx\right]$$

$$= -\left[\frac{F_{BD}}{3EI}\left(\frac{L}{6}\right)^3 + \frac{F_{BD}}{EI}\int_{0.5L}^{L}\left(x-\frac{L}{3}\right)^2 dx + \frac{w_0}{2EI}\int_{0.5L}^{L}\left(x-\frac{L}{2}\right)^2\left(x-\frac{L}{2}+\frac{L}{6}\right)dx\right]$$

$$= -\left[\frac{F_{BD}}{3EI}\left(\frac{L}{6}\right)^3 + \frac{7F_{BD}L^3}{72EI} + \frac{w_0}{2EI}\int_{0.5L}^{L}\left(x-\frac{L}{2}\right)^3 dx + \frac{w_0}{2EI}\int_{0.5L}^{L}\left(x-\frac{L}{2}\right)^2\frac{L}{6}dx\right]$$

$$= -\left(\frac{8F_{BD}L^3}{81EI} + \frac{13w_0L^4}{1152EI}\right)$$

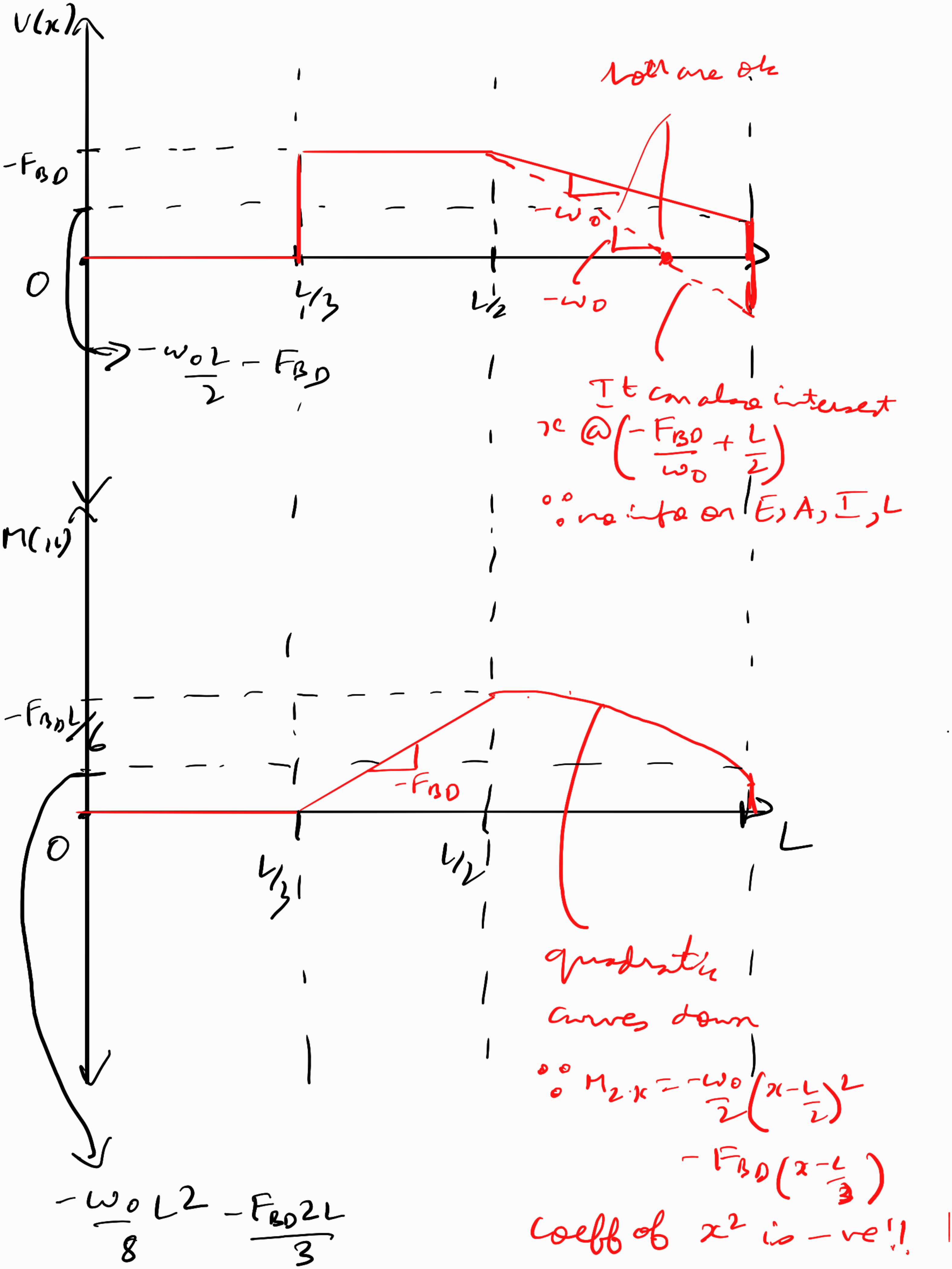
$$v_B = e_{BD} = \frac{F_{BD}(L/2)}{2EA}$$ extension of rod F_{BD} in +ve y ∴ extension

Thus,

$$-\left(\frac{8F_{BD}L^3}{81EI} + \frac{13w_0L^4}{1152EI}\right) = \frac{F_{BD}(L/2)}{2EA}$$

$$\Rightarrow F_{BD} = \frac{-13w_0L^3}{1152I} / \left(\frac{1}{4A} + \frac{8L^2}{81I}\right)$$

$$\Rightarrow v_B = \frac{F_{BD}(L/2)}{2EA} = \frac{-117I}{128(81I+32AL^2)}\frac{w_0L^4}{EI}$$



V(x)
both are ok
-F_BD
O
L/3
L/2
-w_0
-w_0
-w_0 L/2 - F_BD
It can also intersect
x @ (-F_BD/w_0 + L/2)
∴ no info on E, A, I, L
M(x)
-F_BD L/6
-F_BD
O
L/3
L/2
L
quadratic
curves down
∴ M_2 x = -w_0/2 (x - L/2)^2
- F_BD (x - L/3)
coeff of x^2 is -ve!!
-w_0 L^2/8 - F_BD 2L/3

Problem 2 (10 points):

A three-segment rod AD is fixed to walls at A and D. An external load $2P$ is applied at C. The properties are shown in the figure.

(1) Use three finite elements (one element per segment), write down the stiffness matrix K and the forcing vector F.

(2) Enforcing the boundary conditions, write the reduced system of equations and solve for the displacements at B and C.

(3) Find the reactions at A and D.

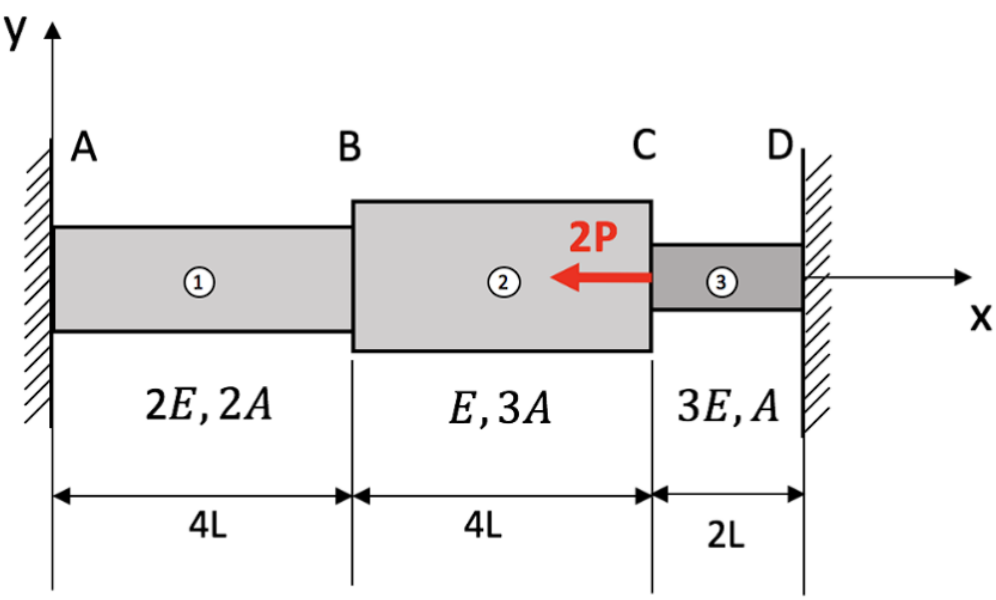


Figure 2: 3 rod structure for FEA in Problem 2 (spring 2020)

Solution:

$$k_1 = \frac{2E * 2A}{4L} = \frac{EA}{L}$$

$$k_2 = \frac{E * 3A}{4L} = \frac{3EA}{4L}$$

$$k_3 = \frac{3E * A}{2L} = \frac{3EA}{2L}$$

Stiffness matrix:

$$[K] = \begin{bmatrix} k_1 & -k_1 & & \\ k_1 & k_1 + k_2 & -k_2 & \\ & -k_2 & k_2 + k_3 & -k_3 \\ & & -k_3 & k_3 \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & & \\ -1 & 1.75 & -0.75 & \\ & -0.75 & 2.25 & -1.5 \\ & & -1.5 & 1.5 \end{bmatrix}$$

Forcing vector:

$$[F] = \begin{bmatrix} F_A \\ F_B \\ F_C \\ F_D \end{bmatrix} = \begin{bmatrix} F_A \\ 0 \\ -2P \\ F_D \end{bmatrix}$$

Displacements (using boundary conditions $u_A = u_D = 0$)

$$[u] = \begin{bmatrix} u_A \\ u_B \\ u_C \\ u_D \end{bmatrix} = \begin{bmatrix} 0 \\ u_B \\ u_C \\ 0 \end{bmatrix}$$

$$[F] = [K][u]$$

$$\begin{bmatrix} F_A \\ 0 \\ -2P \\ F_D \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1 & -1 & & \\ -1 & 1.75 & -0.75 & \\ & -0.75 & 2.25 & -1.5 \\ & & -1.5 & 1.5 \end{bmatrix}\begin{bmatrix} 0 \\ u_B \\ u_C \\ 0 \end{bmatrix}$$

Reduced system of equations:

$$\begin{bmatrix} 0 \\ -2P \end{bmatrix} = \frac{EA}{L}\begin{bmatrix} 1.75 & -0.75 \\ -0.75 & 2.25 \end{bmatrix}\begin{bmatrix} u_B \\ u_C \end{bmatrix}$$

solving for displacements at B and C:

$$u_B = -\frac{4}{9}\frac{PL}{EA}$$

$$u_c = -\frac{28}{27}\frac{PL}{EA}$$

Reactions at A and D:

$$F_A = \left(-\frac{EA}{L}\right) * \left(-\frac{4}{9}\frac{PL}{EA}\right) = \frac{4}{9}P$$

$$F_D = \left(-\frac{3EA}{2L}\right) * \left(-\frac{28}{27}\frac{PL}{EA}\right) = \frac{14}{9}P$$

Problem 3 (10 points):

Problem 8.3 (10 points) Two segments, AB and BC, with a thin walled hollow circular cross section of outer diameter a and inner diameter 0.8a, are welded together at B to form the L-shaped frame ABC shown in the figure below.

Use the following data in your analysis: E = 280 GPa, G = 120 GPa, a = 20 mm, M0 =1000 Nm

(a) Reaction at B

(b) Slope θ at point A.

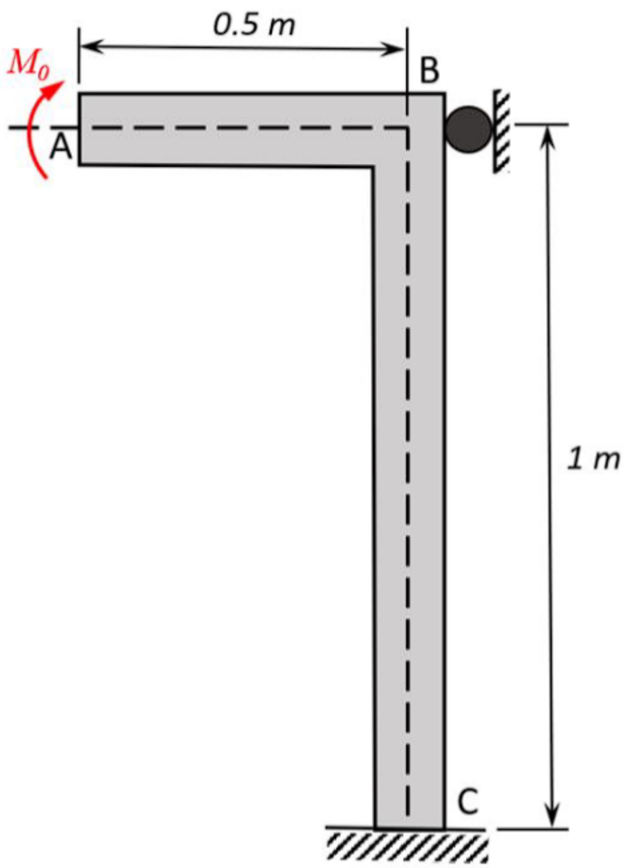
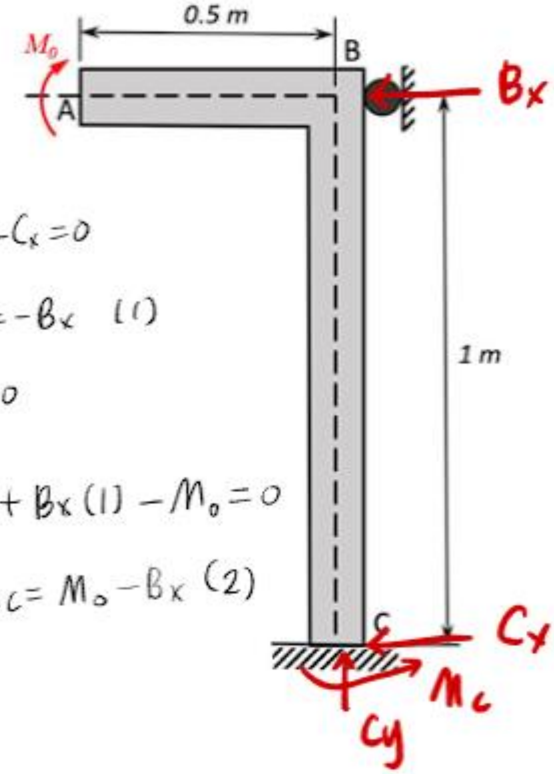


Figure 3: L for problem 3

(c) [redacted]

(d) [redacted]

(a)



3D schematic

$\Sigma F_x: \; -B_x - C_x = 0$

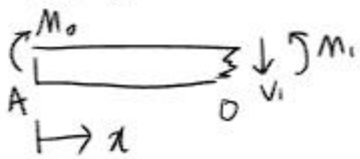
$C_x = -B_x \quad (1)$

$\Sigma F_y: \; C_y = 0$

$\Sigma M_c: \; M_c + B_x(1) - M_0 = 0$

$\Rightarrow M_c = M_0 - B_x \quad (2)$

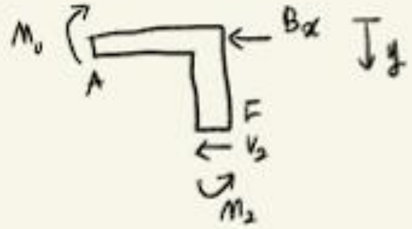
Section AB



$\Sigma F_y: \; -V_1 = 0$

$\Sigma M_O: \; M_1 - M_0 = 0 \rightarrow M_1 = M_0$

Section BC



$\Sigma F_x: \; -B_x - V_2 = 0$

$\rightarrow V_2 = -B_x$

$\Sigma M_F: \; M_2 + B_x y - M_0 = 0$

$\rightarrow M_2 = M_0 - B_x y$

$$U = U_\sigma + U_\tau = U_{\sigma_1} + U_{\sigma_2} + \cancel{U_{\tau_1}} + U_{\tau_2} \quad (V_1 = 0)$$

$$= \frac{1}{2EI}\left\{ \int_0^{0.5} M_1^2\, dx + \int_0^1 M_2^2\, dy \right\} + \frac{f_s}{2GA}\int_0^1 V_2^2\, dy$$

$$= \frac{1}{2EI}\left\{ [M_0^2 x]_0^{0.5} + \int_0^1 (M_0 - B_x y)^2\, dy + \frac{f_s}{2GA}\int_0^1 B_x^2\, dy$$

$$\frac{\partial U}{\partial B_x} = 0$$

$$\Rightarrow \frac{1}{2EI}\int_0^1 2(M_0 - B_x y)(-y)\, dy + \frac{f_s}{GA}\int_0^1 B_x\, dy$$

$$= -\frac{1}{EI}\left(M_0 \frac{y^2}{2} - \frac{B_x}{3} y^3 \right]_0^1 + \frac{f_s}{GA} B_x = 0$$

$$\Rightarrow \frac{1}{EI}\left(\frac{M_0}{2} - \frac{B_x}{3} \right) = \frac{f_s B_x}{GA}$$

$$\Rightarrow B_x\left(\frac{f_s}{GA} + \frac{1}{3EI} \right) = \frac{M_0}{2EI}$$

$$\Rightarrow B_x = \frac{M_o}{2EI}\left\{\frac{f_s}{GA} + \frac{1}{3EI}\right\}^{-1}$$

$f_s = 2$ (thin walled tube) $A = \frac{\pi}{4}(D^2 - d^2) = 0.36\pi \times 10^{-4} \, m^2$

$I = \frac{\pi}{64}(D^4 - d^4) = 1.476\pi \times 10^{-9} \, m^4$

$$\therefore B_x = 0.385\left\{14.736 \times 10^{-8} + 2.567 \times 10^{-4}\right\}^{-1}$$

$B_x = 1.499 \, kN$

From eqns ① & ②

$C_y = -1.499 \, kN$ $M_c = -499 \, Nm$

b) $\theta_x = \frac{\partial U}{\partial M_o} = \frac{1}{2EI}\left\{\int_0^{0.5} 2M_o \, dx + 2\int_0^1 (M_o - B_x y) \, dy\right\}$

$$= \frac{1}{2EI}\left\{M_o + 2\left[M_o y - \frac{B_x y^2}{2}\right]_0^1\right\}$$

$$= \frac{1}{2EI}\left\{3M_o - B_x\right\}$$

$$= 3.851 \times 10^{-4}\{1501\}$$

$$= 0.578 \text{ rad} = 33.12^\circ$$

(along the dir of M_o)

Problem 4 (2.5 points + 2.5 points):

4.1 Considering the stepped shaft below with the ends fixed. Torsion is applied at B. The shear modulus of shafts 1,2,3 are 2G, 3G and G respectively. The polar moment of inertia of 1,2,3 are 6I, 2I, 12I. If you were to use 3 finite elements to solve this scenario what would the rotation stiffness matrix K_ϕ look like?

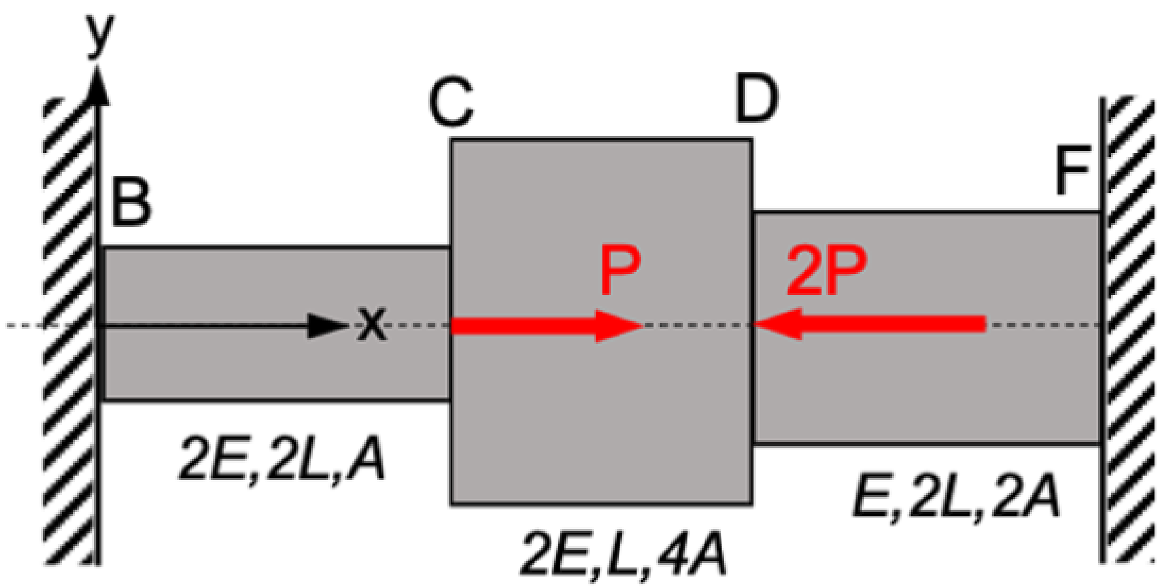
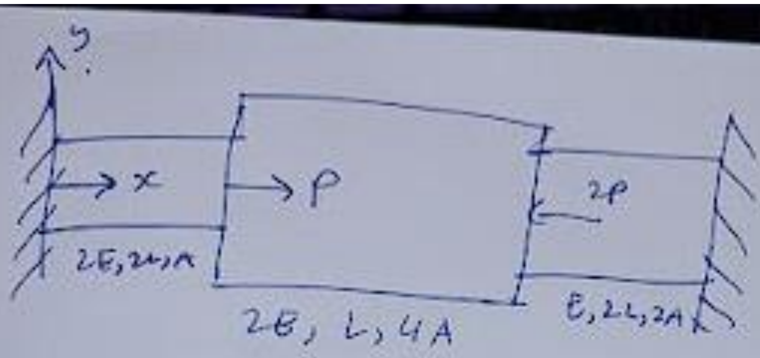


Figure 4.1: Stepped shaft in torsion case for problem 4.1(Spring 20)



4 nodes ∴ 4×4 elements a) and b)

$$k_1 = \frac{2E \times A}{2L} = \frac{EA}{L}$$

$$k_2 = \frac{2E \times 4A}{L} = 8\frac{EA}{L}$$

$$k_3 = \frac{E \times 2A}{2L} = \frac{EA}{L}$$

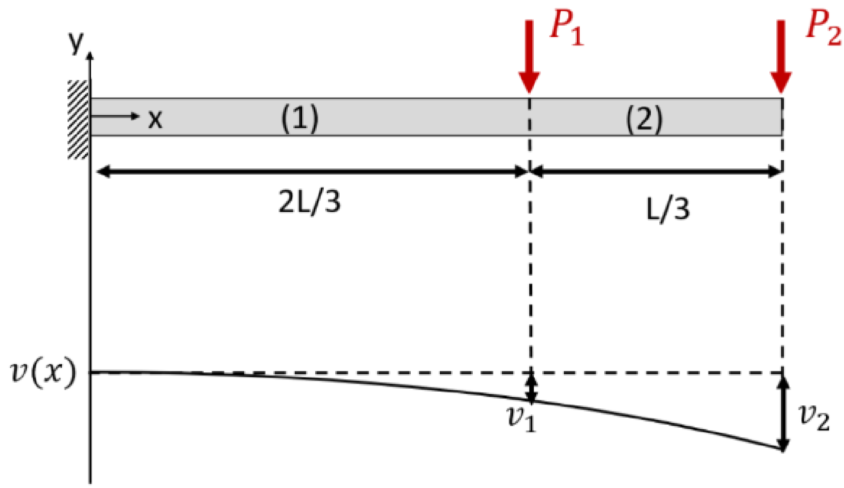
~~$$k = \frac{EA}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 9 & -8 & 0 \\ 0 & -8 & 10 & -1 \\ 0 & 0 & -1 & 1.0 \end{bmatrix}$$~~

$$K = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 9 & -8 & 0 \\ 0 & -8 & 9 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

c)

4.2

4.2 A cantilever beam has 2 loads P_1 and P_2 acting on it as shown below:



If we were to apply Castigliano's theorem to obtain deflection v_1, what would the equation look like (ignore shear energy):

a) $v_1 = \frac{\delta U}{\delta P_2} = \frac{1}{EI}\int_0^L M_1 \left(\frac{\delta M_1}{\delta P_2}\right) dx + \frac{1}{EI}\int_0^L M_2 \left(\frac{\delta M_2}{\delta P_2}\right) dx$

b) $v_1 = \frac{\delta U}{\delta P_1} = \frac{1}{EI}\int_0^{\frac{2L}{3}} M_1 \left(\frac{\delta M_1}{\delta P_1}\right) dx + \frac{1}{EI}\int_{\frac{2L}{3}}^L M_2 \left(\frac{\delta M_2}{\delta P_1}\right) dx$

c) $v_1 = \frac{\delta U}{\delta P_1} = \frac{1}{EI}\int_0^{\frac{2L}{3}} M_1 \left(\frac{\delta M_1}{\delta P_1}\right) dx + \frac{1}{EI}\int_0^L M_2 \left(\frac{\delta M_2}{\delta P_1}\right) dx$

d) $v_1 = \frac{\delta U}{\delta P_2} = \frac{1}{EI}\int_0^{\frac{2L}{3}} M_1 \left(\frac{\delta M_1}{\delta P_2}\right) dx + \frac{1}{EI}\int_{\frac{2L}{3}}^L M_2 \left(\frac{\delta M_2}{\delta P_2}\right) dx$

e) None of the above

b) is the correct answer

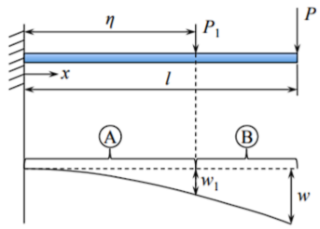


Figure 7.6.5: Cantilever beam loaded b two point forces.

Illustration

Consider a cantilever beam loaded by two point forces. One force P is applied at the tip and the other force P_1 acts at a distance η from the support.

The bending moment distribution is

$$\begin{aligned} M_A(x) &= P(l-x) + P_1(\eta - x) \quad &\text{for} \quad 0 < x < \eta \\ M_B(x) &= P(l-x) \quad &\text{for} \quad \eta < x < l \end{aligned} \tag{7.6.15}$$

The bending strain energy is

$$U(P, P_1) = \frac{1}{2EI}\int_0^\eta M_A^2 dx + \frac{1}{2EI}\int_\eta^l M_B^2 dx \tag{7.6.16}$$

According to Equation 7.6.3, the deflection under the point load P_1 is

$$w_1 = \frac{\partial U(P, P_1)}{\partial P_1} = \frac{1}{EI}\int_0^\eta M_A \frac{\partial M_A}{\partial P_1} dx + \frac{1}{EI}\int_\eta^l M_B \frac{\partial M_B}{\partial P_1} dx \tag{7.6.17}$$