Chapter II

Free response of discrete systems

Introduction

In the last chapter of lecture notes, we saw that small motion corresponding to the free response of an N-DOF discrete system can be described by the following set of N differential equations:

$$[M]\,\ddot{\vec{x}} + [C]\,\dot{\vec{x}} + [K]\,\vec{x} = \vec{0}$$

where $[M]$, $[C]$ and $[K]$ are the $(N \times N)$ mass, damping and stiffness matrices, respectively. Since these are linear, constant-coefficient, homogeneous differential equations, the general form of their solution can be written as the following time-dependent vector:

$$\vec{x}(t) = \vec{X}e^{\lambda t}$$

where, at this point $\vec{X}$ and λ are unknowns. Substituting this solution form into the homogeneous differential equations produces the following:

$$\left[\lambda^2\,[M] + \lambda\,[C] + [K]\right]\vec{X}e^{\lambda t} = \vec{0}$$

Note that $e^{\lambda t} \neq 0$ for any finite real or complex value of λt. Therefore, the above reduces to the following set of N algebraic equations:

$$\left[\lambda^2\,[M] + \lambda\,[C] + [K]\right]\vec{X} = \vec{0}$$

These equations represent an "eigenvalue problem" whose solutions λ and $\vec{X}$ are known as the eigenvalues and eigenvectors for the free vibration problem.

In this chapter of the notes we will study the above eigenvalue problem and how the corresponding eigenvalues and eigenvectors are used in the construction of the free vibration response of the system. The properties of the solution of the eigenvalue problem will aid us not only in constructing the solutions, but also in interpreting the nature of the response.

II.1 Free response: single-DOF systems

In this section of the notes, we will study the free response of single-DOF systems governed by the following linear, 2nd order differential equation:

$$m\ddot{x} + c\dot{x} + kx = 0$$

As seen in the Introduction to this chapter, the assumed solution form $x(t) = Xe^{\lambda t}$ produces the following eigenvalue problem for $N = 1$:

$$\left[\lambda^2 m + \lambda c + k\right] X = 0$$

Note that $X = 0$ satisfies the above equation. However, $X = 0$ implies that $x(t) = 0$. This corresponds to no motion of the system (a "trivial" solution) and is not of interest to us. For $X \neq 0$, the eigenvalue problem becomes:

$$\lambda^2 m + \lambda c + k = 0$$

Since the above is true for any non-zero value of X, we can arbitrary choose a non-zero value for X. For convenience, we will choose $X = 1$.

The above is known as the "characteristic equation" for the system. Since this is a quadratic equation in λ, it has two solutions (the "eigenvalues") given by the following:

$$\begin{aligned}
\lambda_{1,2} &= \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \\
&= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \\
&= -\frac{c}{2\sqrt{km}}\sqrt{\frac{k}{m}} \pm \sqrt{\frac{k}{m}}\sqrt{\left(\frac{c}{2\sqrt{km}}\right)^2 - 1} \\
&= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}
\end{aligned}$$

where we have introduced the following two parameters:

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

The parameter ζ defined above (known as the "damping ratio") dictates the character of the eigenvalues: i) if $0 \leq \zeta < 1$, then the two eigenvalues are COMPLEX and distinct (and in complex conjugate pairs); ii) if $\zeta = 1$, then the two eigenvalues are REAL and repeated; and, iii) if $\zeta > 1$, then the two eigenvalues are REAL and distinct.

If the eigenvalues are distinct (as is true for $\zeta \neq 1$), then the two solutions $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ are independent. Therefore, for this case, the total free vibration response can be written as a linear combination of $x_1(t)$ and $x_2(t)$:

$$\begin{aligned} x(t) &= ax_1(t) + bx_2(t) \\ &= ae^{\lambda_1 t} + be^{\lambda_2 t} \end{aligned}$$

Recall that for $\zeta = 1$, the eigenvalues are repeated, and, as a result, the solutions $x_1(t) = Xe^{\lambda_1 t}$ and $x_2(t) = Xe^{\lambda_2 t}$ are NOT independent. Because of this, we will need to treat this as a special case, as will be seen later on.

- $0 \leq \zeta < 1$ **(underdamped response)**. As discussed above, the eigenvalues appear in complex conjugate pair:

  $$\begin{aligned} \lambda_{1,2} &= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} \\ &= -\zeta\omega_n \pm i\omega_d \end{aligned}$$

  where $i = \sqrt{-1}$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

  Since the eigenvalues are distinct, the total free vibration solution is written as:

  $$\begin{aligned} x(t) &= ae^{\lambda_1 t} + be^{\lambda_2 t} \\ &= ae^{(-\zeta\omega_n - i\omega_d)t} + be^{(-\zeta\omega_n + i\omega_d)t} \\ &= e^{-\zeta\omega_n t}\left[ae^{-i\omega_d t} + be^{+i\omega_d t}\right] \\ &= e^{-\zeta\omega_n t}\left[a\left(cos\omega_d t - isin\omega_d t\right) + b\left(cos\omega_d t + isin\omega_d t\right)\right] \\ &= e^{-\zeta\omega_n t}\left[(a+b)\,cos\omega_d t + i\left(-a+b\right)sin\omega_d t\right] \\ &= e^{-\zeta\omega_n t}\left[Ccos\omega_d t + Ssin\omega_d t\right] \end{aligned}$$

  where $C = a + b$ and $S = i\left(-a+b\right)$.

- $\zeta > 1$ **(overdamped response)**. As discussed above, the eigenvalues are real and distinct:

  $$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

  Since the eigenvalues are distinct, the total free vibration solution is written as:

  $$\begin{aligned} x(t) &= ae^{\lambda_1 t} + be^{\lambda_2 t} \\ &= ae^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t} + be^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} \end{aligned}$$

- $\zeta > 1$ **(critically response)**. As discussed above, the eigenvalues are real and repeated. Since the roots are repeated, the solutions $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ are not independent, and, as a result:

  $$x(t) \neq ae^{\lambda_1 t} + be^{\lambda_2 t}$$

  What do we do in this case??

Enforcing initial conditions (ICs)

The free response for a single-DOF system:

$$m\ddot{x} + c\dot{x} + kx = 0$$

can be written in the general form of:

$$x(t) = au(t) + bv(t)$$

where the form of the fundamental solution pairs, $u(t)$ and $v(t)$ are given in the following table as a function of the damping ratio ζ. Recall the following definitions for parameters appearing in this table:

$$\zeta = \frac{c}{2\sqrt{km}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

damping ratio	***u(t)***	***v(t)***
$\zeta = 0$	$\cos\omega_n t$	$\sin\omega_n t$
$0 < \zeta < 1$	$e^{-\zeta\omega_n t}\cos\omega_d t$	$e^{-\zeta\omega_n t}\sin\omega_d t$
$\zeta = 1$	$e^{-\zeta\omega_n t}$	$t\,e^{-\zeta\omega_n t}$
$\zeta > 1$	$e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t}$	$e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$

The coefficients a and b are found by enforcing initial conditions of $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$ on the solution form:

$$x(0) = x_0 = au(0) + bv(0)$$

$$\dot{x}(0) = \dot{x}_0 = a\dot{u}(0) + b\dot{v}(0)$$

Solving the above algebraic equations for a and b gives:

$$a = \frac{\dot{v}(0)x_0 - v(0)\dot{x}_0}{u(0)\dot{v}(0) - v(0)\dot{u}(0)}$$

$$b = \frac{-\dot{u}(0)x_0 + u(0)\dot{x}_0}{u(0)\dot{v}(0) - v(0)\dot{u}(0)}$$

Consideration of underdamped response: estimation of damping ratio

Recall that the general form of underdamped $\zeta < 1$ response is given by:

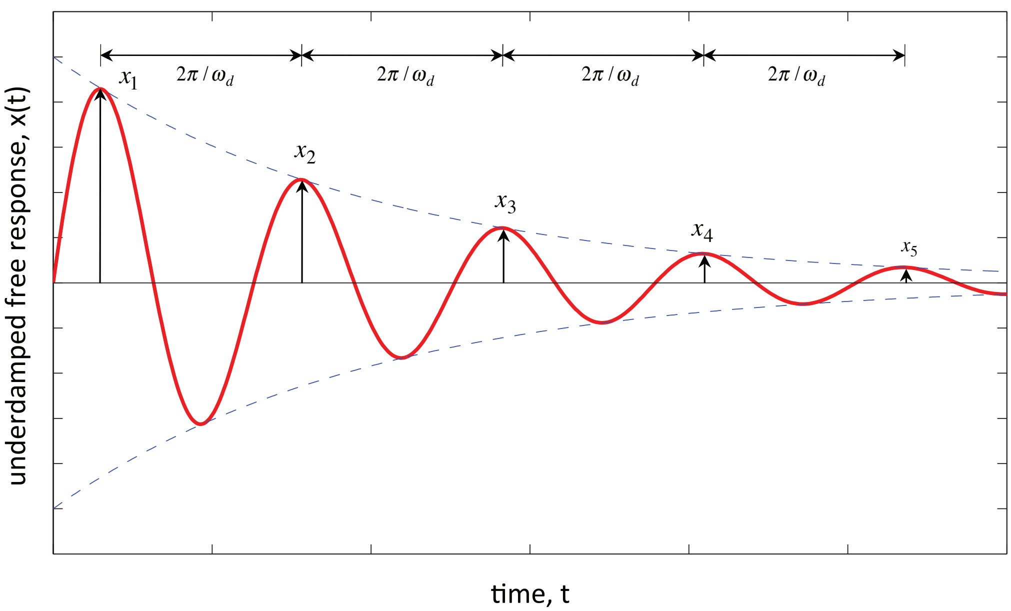
$$x(t) = e^{-\zeta\omega_n t}\left[C cos\omega_d t + S sin\omega_d t\right]$$

where C and S are determined by the ICs. This solution can alternately be written in terms of an amplitude X and phase ϕ as:

$$x(t) = X e^{-\zeta\omega_n t} sin\left(\omega_d t + \phi\right)$$

where $X = \sqrt{C^2 + S^2}$ and $\phi = tan^{-1}\left(\frac{C}{S}\right)$ depend on the ICs through C and S.

From this solution form, we see that the free response is made up of an oscillatory component (having a frequency of $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and phase shifted by ϕ) that is amplitude modulated by a decaying exponential (whose decay rate is related to $\zeta\omega_n$). A sketch of the response corresponding to $\zeta = 0.1$, $x(0) = 0$ and $\dot{x}(0) > 0$ is shown below.



Here we will explore an method by which we can estimate the damping ratio from an underdamped response plot. To this end, let x_1, x_2, x_3, ... represent successive local maximum values of x(t) occurring at times t_1, t_2, t_3, ... (as shown in the figure). These local maxima of $x(t)$ occur when:

$$\dot{x}(t_j) = X e^{-\zeta\omega_n t_j}\left[-\zeta\omega_n sin\left(\omega_d t_j + \phi\right) + \omega_d cos\left(\omega_d t_j + \phi\right)\right] = 0$$

From this, we see that:

$$tan\left(\omega_d t_j + \phi\right) = \frac{sin\left(\omega_d t_j + \phi\right)}{cos\left(\omega_d t_j + \phi\right)} = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

From this, we see that the local maxima occur at equally-spaced time intervals of:

$$t_{j+1} - t_j = \frac{2\pi}{\omega_d}$$

Therefore,

$$\begin{aligned}\frac{x_j}{x_{j+1}} &= \frac{Xe^{-\zeta\omega_n t_j} sin\left(\omega_d t_j + \phi\right)}{Xe^{-\zeta\omega_n t_{j+1}} sin\left(\omega_d t_{j+1} + \phi\right)} \\ &= e^{\zeta\omega_n\left(t_{j+1} - t_j\right)} \\ &= e^{\zeta\omega_n\left(2\pi/\omega_d\right)} \\ &= e^{\zeta\omega_n\left(2\pi/\omega_n\sqrt{1-\zeta^2}\right)} \\ &= e^{2\pi\zeta/\sqrt{1-\zeta^2}}\end{aligned}$$

Therefore, we have:

$$ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

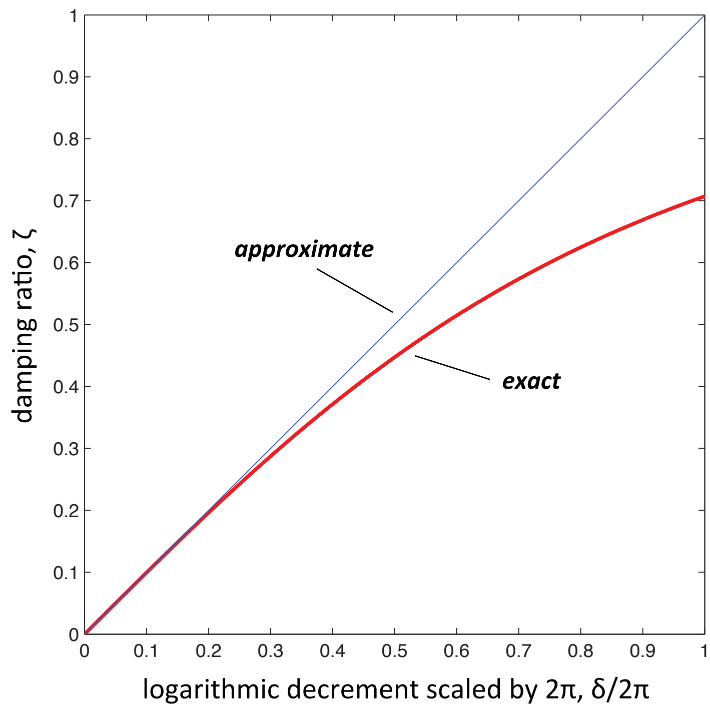
or, solving for ζ:

$$\zeta = \frac{ln\left(x_j/x_{j+1}\right)}{\sqrt{4\pi^2 + ln^2\left(x_j/x_{j+1}\right)}} = \frac{\delta/2\pi}{\sqrt{1+\left(\delta/2\pi\right)^2}}$$

where $\delta = ln\left(x_j/x_{j+1}\right)$ = "logarithmic decrement" of the response. For small damping ($\zeta << 1$), the above expression damping ratio in terms of the logarithmic decrement reduces to:

$$\zeta = \frac{\delta}{2\pi}$$

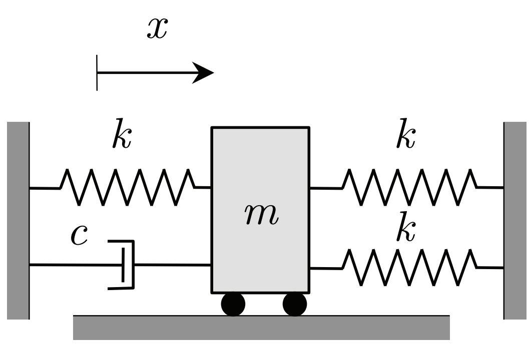
A comparison of the exact and approximate representations of damping ratio ζ and logarithmic decrement δ are shown below. It is seen that the approximate relation is quite accurate for damping ratios up to around 0.4.



approximate
exact
damping ratio, ζ
logarithmic decrement scaled by 2π, δ/2π
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

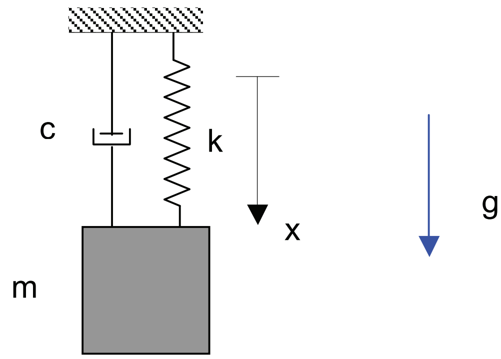
Example II.1.1

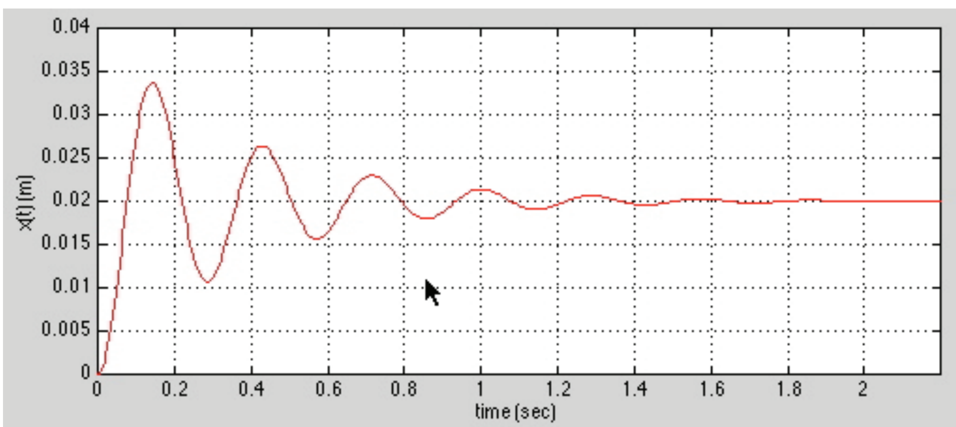
Find the response $x(t)$ of the system shown below for $m = 3$ kg, $c = 9$ kg/sec and $k = 2500$ N/m. Use initial conditions of: $x_0 = 0.1$m and $\dot{x}_0 = 2$m/sec.



Example II.1.2

A particle of mass m is connected to ground with a linear spring and dashpot, as shown. When $x = 0$, the spring is unstretched. The particle is released from rest giving a time history of response that is shown below (times and values of local maxima of $x(\mathrm{t})$ are summarized in the following table). If $m = 1$ kg, find the stiffness of the spring k and the damping constant c of the dashpot.





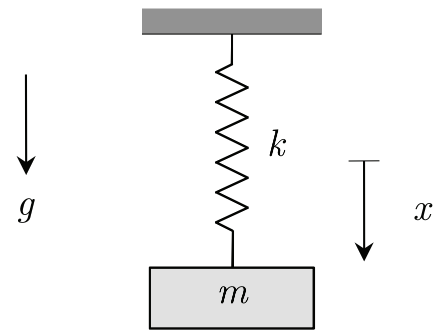
j	t_j (secs)	x_j (mm)
1	0.140	33.6
2	0.426	26.3
3	0.712	22.9
4	1.002	21.4

Example II.1.3

A particle of mass m is supported by a spring, as shown. When $x = 0$, the spring is unstretched. The restoring force in the spring is given by the cubic (nonlinear) expression:

$$F_s = -kx - \beta x^3$$

For $k = 500$ N/m and $m = 1$ kg, the stretch in the spring is known at static equilibrium is known to be 10 mm. Find the natural frequency for small oscillations about the equilibrium state.



II.2 Free response: undamped multi-DOF systems

In this section of the notes, we will study the undamped free response of multi-DOF systems governed by the following set of linear, 2nd order differential equations:

$$[M]\,\ddot{\vec{x}} + [K]\,\vec{x} = \vec{0}$$

As seen in the Introduction to this chapter, the assumed solution form $\vec{x}(t) = \vec{X}e^{\lambda t}$ produces the following eigenvalue problem:

$$\left[\lambda^2\,[M] + [K]\right]\vec{X} = \vec{0}$$

Note that $\vec{X} = \vec{0}$ satisfies the above equation. However, $\vec{X} = \vec{0}$ implies that $\vec{x}(t) = 0$. This corresponds to no motion of the system (a "trivial" solution) and is not of interest to us. For $\vec{X} \neq 0$, the matrix $\left[\lambda^2\,[M] + [K]\right]$ must be singular:

$$det\left[\lambda^2\,[M] + [K]\right] = 0$$

Expanding the above determinant produces an Nth-order polynomial in λ^2:

$$b_N\lambda^{2N} + b_{N-1}\lambda^{2(N-1)} + b_{N-2}\lambda^{2(N-2)}... + b_1\lambda^2 + b_0 = 0$$

where the coefficients $b_0, b_1, ..., b_N$ depend on the elements of $[M]$ and $[K]$. The above polynomial is known as the "characteristic equation" for the system. The characteristic equation will have N roots for λ^2, and therefore, $2N$ roots for λ (known as the "eigenvalues" of the system).

For each root λ_j^2 $(j = 1, 2, ..., N)$ of the characteristic equation, we can solve for the corresponding vector $\vec{X}^{(j)}$ $(j = 1, 2, ..., N)$ using the following equations:

$$\left[\lambda_j^2\,[M] + [K]\right]\vec{X}^{(j)} = \vec{0}$$

This produces N vectors $\vec{X}^{(j)}$ $(j = 1, 2, ..., N)$ known as the "eigenvectors" (or, "modal vectors") for the system.

IMPORTANT NOTE: For a given eigenvalue, you will find that there is not a unique eigenvector. To see this, suppose that you have found eigenvector $\vec{X}^{(j)}$ from the above equations. Any non-zero multiple of this eigenvector $\alpha\vec{X}^{(j)}$ also satisfies this equation:

$$\left[\lambda_j^2\,[M] + [K]\right]\left(\alpha\vec{X}^{(j)}\right) = \vec{0}$$

Because of this, we are free to choose any scale factor α that we want. Typically this scale is set by choosing one of the components of the vector $\vec{X}^{(j)}$ to be unity ('1'), and solve for the remaining $N-1$ components. Later on, we will discuss an alternate way of scaling these eigenvectors.

Some important properties of the roots of the undamped characteristic equation

Recall that the Lagrangian formulation produces symmetric mass and stiffness matrices: $[M] = [M]^T$ and $[K] = [K]^T$.

- If the matrices $[M]$ and $[K]$ are symmetric, then the roots λ_j^2 $(j = 1, 2, ..., N)$ will all be REAL.
- If the roots λ_j^2 $(j = 1, 2, ..., N$ are all real, then all of the eigenvectors $\vec{X}^{(j)}$ $(j = 1, 2, ..., N)$ will also be real. Furthermore, with symmetric mass and stiffness matrices, the eigenvectors will form an independent set of vectors; that is, no one vector can be written as a linear combination of the remaining vectors.
- If the matrices $[M]$ and $[K]$ are both "positive definite", then all of the roots λ_j^2 $(j = 1, 2, ..., N$ will be negative: $\lambda_j^2 = -\omega_j^2$. Therefore, all the roots λ_j $(j = 1, 2, ..., 2N$ will be purely imaginary:

  $$\lambda_j = \pm i\omega_j$$

  where ω_j $(j = 1, 2, ..., 2N)$ are real. Recognizing this, we will often write the eigenvalue problem for undamped systems as:

  $$\left[-\omega_j^2 [M] + [K]\right] \vec{X}^{(j)} = \vec{0}$$

  In general, the mass and stiffness matrices for the systems of interest to us in the course will be positive definite. In Appendix II, we see how to verify the positive definiteness of your mass and stiffness matrices.

Free response of undamped multi-DOF discrete systems

Since the eigenvectors $\vec{X}^{(j)}$ $(j = 1, 2, ..., N)$ form an independent set, the solutions $\vec{X}^{(j)}e^{\pm\omega_j t}$ $(j = 1, 2, ..., N)$ are independent. From this, we can say that the total free response is a linear combination of these solutions:

$$\begin{aligned}
\vec{x}(t) &= \sum_{j=1}^{N} \vec{X}^{(j)} \left[a_j e^{-i\omega_j t} + b_j e^{i\omega_j t}\right] \\
&= \sum_{j=1}^{N} \vec{X}^{(j)} \left[a_j \left(cos\omega_j t - isin\omega_j t\right) + b_j \left(cos\omega_j t + isin\omega_j t\right)\right] \\
&= \sum_{j=1}^{N} \vec{X}^{(j)} \left[\left(a_j + b_j\right) cos\omega_j t + i\left(-a_j + b_j\right) sin\omega_j t\right] \\
&= \sum_{j=1}^{N} \vec{X}^{(j)} \left[c_j cos\omega_j t + s_j sin\omega_j t\right]
\end{aligned}$$

where $c_j = a_j + b_j$ and $s_j = i\left(-a_j + b_j\right)$.

The coefficients c_j and s_j $(j = 1, 2, ..., N)$ are found from imposing the initial conditions on the system:

$$\vec{x}(0) = \sum_{j=1}^{N} \vec{X}^{(j)} c_j$$

$$\dot{\vec{x}}(0) = \sum_{j=1}^{N} \vec{X}^{(j)} \omega_j s_j$$

and solving. This process of solving two sets of N algebraic equations can become a bit tedious if working by hand. In the following we will consider a special property of the eigenvectors that allows us to simplify the solution process, as well as allowing for a clearer interpretation of the results.

Modal vectors orthogonality properties and an application of these properties

The eigenvalue problem relating the jth natural frequency and jth modal vector for an undamped system can be written as:

$$\omega_j^2 [M] \vec{X}^{(j)} = [K] \vec{X}^{(j)}$$

Similarly, for the kth natural frequency and kth modal vector pair, we also have:

$$\omega_k^2 [M] \vec{X}^{(k)} = [K] \vec{X}^{(k)}$$

Say we premultiply the first equation by $\vec{X}^{(k)T}$, premultiply the second equation by $\vec{X}^{(j)T}$ and subtract the results:

$$\omega_j^2 \vec{X}^{(k)T} [M] \vec{X}^{(j)} - \omega_k^2 \vec{X}^{(j)T} [M] \vec{X}^{(k)} = \vec{X}^{(k)T} [K] \vec{X}^{(j)} - \vec{X}^{(j)T} [K] \vec{X}^{(k)}$$

[Aside: Consider a $N \times N$ matrix $[A]$ and two N-vectors $\vec{a}$ and $\vec{b}$. In general, $\vec{a}^T [A] \vec{b} \neq \vec{b}^T [A] \vec{a}$ (i.e., the order of multiplication is not reversible). However, if $[A]$ is SYMMETRIC, then $\vec{a}^T [A] \vec{b} = \vec{b}^T [A] \vec{a}$.]

Therefore, the above equation becomes:

$$\begin{aligned} \omega_j^2 \vec{X}^{(j)T} [M] \vec{X}^{(k)} - \omega_k^2 \vec{X}^{(j)T} [M] \vec{X}^{(k)} &= \vec{X}^{(j)T} [K] \vec{X}^{(k)} - \vec{X}^{(j)T} [K] \vec{X}^{(k)} \\ \left(\omega_j^2 - \omega_k^2\right) \vec{X}^{(j)T} [M] \vec{X}^{(k)} &= 0 \end{aligned}$$

Up to this point, we have not made any assumptions regarding the roots of the characteristic equation. We know that for positive definite, symmetric mass and stiffness matrices, $[M]$ and $[K]$, the roots for the natural frequencies are all real. It is possible, however, that the roots could be repeated; that is, two or more natural frequencies could be equal. Typically, this is not something that can be determined prior to solving the characteristic equation.

Let's assume at this point that none of the natural frequencies are repeated; that is, $\omega_j \neq \omega_k$. For this case, the above equation provides that:

$$\vec{X}^{(j)T} [M] \vec{X}^{(k)} = 0$$

In words, under the conditions above of non-repeated roots, we see that the modal vectors $\vec{X}^{(j)}$ and $\vec{X}^{(k)}$ are "orthogonal through the mass matrix $[M]$" for $j \neq k$.

It can also be shown that the modal vectors are also orthogonal through the stiffness matrix $[K]$. To show this, repeat the above process, except add together the two results of the vector premultiplications:

$$\begin{aligned} \omega_j^2 \vec{X}^{(j)T} [M] \vec{X}^{(k)} + \omega_k^2 \vec{X}^{(j)T} [M] \vec{X}^{(k)} &= \vec{X}^{(j)T} [K] \vec{X}^{(k)} + \vec{X}^{(j)T} [K] \vec{X}^{(k)} \\ \left(\omega_j^2 + \omega_k^2\right) \vec{X}^{(j)T} [M] \vec{X}^{(k)} &= 2\vec{X}^{(j)T} [K] \vec{X}^{(k)} \end{aligned}$$

Since $\vec{X}^{(j)T}\,[M]\,\vec{X}^{(k)} = 0$ for $j \neq k$, the above shows that:

$$\vec{X}^{(j)T}\,[K]\,\vec{X}^{(k)} = 0$$

Thus, the modal vectors are also orthogonal through the stiffness matrix $[K]$.

Note that the above result was shown to be true for the case of non-repeated roots. What if two or more natural frequencies are equal - does this change that result? It turns out that even for the case of repeated roots of the characteristic equation, the modal vectors are still linearly independent, although not necessarily orthogonal. However, any set of linearly independent vectors can be made orthogonal through a orthogonalization process (e.g., through a Gram-Schmidt orthogonalization). This process is beyond the scope of material covered in this course, and therefore, we will not cover that case here. We will proceed with the understanding that we can always produce a set of orthogonal modal vectors when the mass and stiffness matrices are symmetrical.

Modal vectors normalization and an application of normalization

The eigenvalue problem relating the jth natural frequency and jth modal vector for an undamped system can be written as:

$$\omega_j^2 \left[M\right] \vec{X}^{(j)} = \left[K\right] \vec{X}^{(j)}$$

Say we multiply the above by a non-zero scalar α_j:

$$\omega_j^2 \left[M\right] \left(\alpha_j \vec{X}^{(j)}\right) = \left[K\right] \left(\alpha_j \vec{X}^{(j)}\right)$$

This highlights something that we have discussed before: if $\vec{X}^{(j)}$ is a modal vector for the problem, then so is ANY non-zero scalar multiple of $\vec{X}^{(j)}$. We have seen this in a number of examples that have been worked out in lecture - it was not possible to find a unique vector for any modal vectors. There we generally set the value of one position in the vector to some arbitrary number (such as '1') and then solve for the remaining positions of the vectors.

We are free to scale our modal vectors in any way that we choose. What is the best way? The answer to this depends on the particular application of interest. Here, we will choose the scalar α_j in such a way that ("mass normalization"):

$$\left(\alpha_j \vec{X}^{(j)}\right)^T \left[M\right] \left(\alpha_j \vec{X}^{(j)}\right) = 1$$

Solving for α_j gives:

$$\alpha_j = \frac{1}{\sqrt{\vec{X}^{(j)T} \left[M\right] \vec{X}^{(j)}}}$$

In summary, to produce a set of mass-normalized eigenvectors:

- Solve for the set of eigenvectors from $\left[-\omega_j^2 \left[M\right] + \left[K\right]\right] \vec{X}^{(j)} = \vec{0}$ in the usual way by choosing one position in each eigenvector to be '1'.

- Solve for the scaling factor $\alpha_j = 1/\sqrt{\vec{X}^{(j)T} \left[M\right] \vec{X}^{(j)}}$ for each eigenvector.

- Multiply each eigenvector by its scaling factor to produce a set of normalized eigenvector: $\hat{\vec{X}}^{(j)} = \alpha_j \vec{X}^{(j)}$. Therefore, $\hat{\vec{X}}^{(j)T} \left[M\right] \hat{\vec{X}}^{(j)} = 1$, as desired.

Earlier we saw that the free vibration coefficients c_j and s_j are given by:

$$c_j = \frac{\vec{X}^{(j)T}[M]\vec{x}(0)}{\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$

$$s_j = \frac{1}{\omega_j} \frac{\vec{X}^{(j)T}[M]\dot{\vec{x}}(0)}{\vec{X}^{(j)T}[M]\vec{X}^{(j)}}$$

What are the corresponding relations when we use mass-normalized eigenvectors $\hat{\vec{X}}^{(j)}$?

For mass-normalized modal vectors $\hat{\vec{X}}^{(j)}$, we have, by definition: $\hat{\vec{X}}^{(j)T}[M]\hat{\vec{X}}^{(j)} = 1$. With this, the above equations for the response coefficients c_j and s_j for $j = 1, 2, ...N$ above simplify to the following:

$$c_j = \hat{\vec{X}}^{(j)T}[M]\vec{x}(0)$$

$$s_j = \frac{1}{\omega_j}\hat{\vec{X}}^{(j)T}[M]\dot{\vec{x}}(0)$$

Summary: free response of undamped discrete systems

For the undamped free vibration EOM's of an N-DOF system:

$$[M]\,\ddot{\vec{x}} + [K]\,\vec{x} = \vec{0}$$

with initial conditions of $\vec{x}(0)$ and $\dot{\vec{x}}(0)$, the response has been shown to be:

$$\vec{x}(t) = \sum_{j=1}^{N} \hat{\vec{X}}^{(j)} \left[c_j cos\omega_j t + s_j sin\omega_j t\right]$$

where, for a set of mass-normalized modal vectors:

$$c_j = \hat{\vec{X}}^{(j)T}[M]\vec{x}(0)$$

$$s_j = \frac{1}{\omega_j}\hat{\vec{X}}^{(j)T}[M]\dot{\vec{x}}(0)$$

Remarks:

- From above we see that the total free response is a linear combination of the "modal responses" $\hat{\vec{X}}^{(j)}\left[c_j cos\omega_j t + s_j sin\omega_j t\right]$.
  - The modal responses represent "synchronous" motion (i.e., the shape the motion is preserved for all time) with the shape of the modal response given by the modal vector (hence, modal vectors are often called "mode shapes").
  - The frequencies of the modal responses ω_j (the "natural frequencies") are the roots of the characteristic equation: $det\left[-\omega_j^2\,[M] + [K]\right] = 0$.
  - Note that the size of the response coefficients c_j and s_j depend on the "shape" of the ICs as compared to the shape of the modal vectors. For example, if both sets of ICs have shapes similar to a given modal vector, say the kth mode, then the kth modal response will dominate the response.
- The modal vectors (regardless of normalization) are orthogonal through the symmetric mass and stiffness matrices: $\vec{X}^{(j)T}[M]\vec{X}^{(k)} = 0$ and $\vec{X}^{(j)T}[K]\vec{X}^{(k)} = 0$ for $j \neq k$.
- The simplified form of the response coefficients c_j and s_j above result from the orthogonality of the modal vectors (and, therefore, from the symmetry of $[M]$ and $[K]$). The Lagrangian formulation always produces symmetric mass and stiffness matrices; the Newton-Euler formulation does not. If your Newton-Euler EOM's do not have symmetric mass and stiffness matrices, you will need to use the following equations to solve for the response coefficients:

$$\vec{x}(0) = \sum_{j=1}^{N} \vec{X}^{(j)} c_j$$

$$\dot{\vec{x}}(0) = \sum_{j=1}^{N} \vec{X}^{(j)} \omega_j s_j$$

These are two sets of N coupled algebraic equations whose solution represents considerable computation effort when solved by hand. In addition, an evaluation of modal response contribution to the total response is more difficult. This points to a strong advantage in using the Lagrangian formulation to derive your EOM's!

Special case: response of systems with rigid body modes

We have just seen that the free response of undamped systems is described by the following:

$$\vec{x}(t) = \sum_{j=1}^{N} \hat{\vec{X}}^{(j)} \left[c_j cos\omega_j t + s_j sin\omega_j t\right]$$

where, for a set of mass-normalized modal vectors:

$$c_j = \hat{\vec{X}}^{(j)T}[M]\vec{x}(0)$$

$$s_j = \frac{1}{\omega_j}\hat{\vec{X}}^{(j)T}[M]\dot{\vec{x}}(0)$$

It can be shown that if the stiffness matrix is singular ($det[K] = 0$), then at least one natural frequency of the system is zero. From above, we see that having a zero natural frequency, say $\omega_1 = 0$, produces some complications in calculating the response coefficient s_1 since we are dividing by zero. How do we handle this in our analysis? And, what is the physical interpretation of the response for a system having a zero natural frequency?

For $\omega_1 = 0$:

$$c_1 cos\omega_1 t = c_1$$

$$s_1 sin\omega_1 t = \left(\hat{\vec{X}}^{(j)T}[M]\dot{\vec{x}}(0)\right)\lim_{\omega_1 \to 0}\frac{sin\omega_1 t}{\omega_j} = \left(\hat{\vec{X}}^{(j)T}[M]\dot{\vec{x}}(0)\right)t$$

Therefore, the response with $\omega_1 = 0$ becomes:

$$\vec{x}(t) = (c_1 + s_1 t)\,\hat{\vec{X}}^{(1)} + \sum_{j=2}^{N}\hat{\vec{X}}^{(j)}\left[c_j cos\omega_j t + s_j sin\omega_j t\right]$$

where $s_1 = \hat{\vec{X}}^{(1)T}[M]\dot{\vec{x}}(0)$. From this, it is seen that the modal response corresponding to $\omega_1 = 0$ is not oscillatory; rather, it correspond to rigid motion of the system. The modal vector $\hat{\vec{X}}^{(1)}$ is known as a rigid body mode.

Special case: beating response of systems with nearly equal natural frequencies

Often times when two identical systems are joined together by a weak stiffness coupling, a "beating" response can be observed from the joined system. The beating behavior is characterized by an amplitude modulation of the response and is a direct consequence of the joined system having nearly equal natural frequencies corresponding to characteristically different mode shapes.

Suppose we start out a system from rest ($\dot{\vec{x}}(0) = \vec{0}$). With these ICs, the response can be written as:

$$\vec{x}(t) = \sum_{j=1}^{N} \vec{X}^{(j)} c_j cos\omega_j t$$

Here will separate out the contributions arising from the first two modes from the rest of the response:

$$\begin{aligned}\vec{x}(t) &= \vec{X}^{(1)} c_1 cos\omega_1 t + \vec{X}^{(2)} c_2 cos\omega_2 t + ...\\ &= \frac{1}{2}\left[\vec{X}^{(1)} c_1 - \vec{X}^{(2)} c_2\right](cos\omega_1 t - cos\omega_2 t) + \frac{1}{2}\left[\vec{X}^{(1)} c_1 + \vec{X}^{(2)} c_2\right](cos\omega_1 t + cos\omega_2 t) + ...\end{aligned}$$

Using some trig identities, we see that:

$$cos\omega_1 t + cos\omega_2 t = 2cos\left(\frac{\omega_1 + \omega_2}{2} t\right) cos\left(\frac{\omega_2 - \omega_1}{2} t\right)$$

$$cos\omega_1 t - cos\omega_2 t = -2sin\left(\frac{\omega_1 + \omega_2}{2} t\right) sin\left(\frac{\omega_2 - \omega_1}{2} t\right)$$

Now suppose that $\omega_1 \approx \omega_2$: $\omega_2 = (1 + \epsilon)\,\omega_1$. The response is now given by:

$$\begin{aligned}\vec{x}(t) = -\frac{1}{2}\left[\vec{X}^{(1)} c_1 - \vec{X}^{(2)} c_2\right] sin\,(\omega_1 t)\, sin\left(\frac{\epsilon\omega_1}{2} t\right)\\ + \frac{1}{2}\left[\vec{X}^{(1)} c_1 + \vec{X}^{(2)} c_2\right] cos\,(\omega_1 t)\, cos\left(\frac{\epsilon\omega_1}{2} t\right) + ...\end{aligned}$$

How does the contribution of the first two modes look? To see this, let z_i be the ith component of the response from the first two modes (the ones with nearly identical natural frequencies):

$$\begin{aligned}z_i(t) = -\frac{1}{2}\left[X_i^{(1)} c_1 - X_i^{(2)} c_2\right] sin\,(\omega_1 t)\, sin\left(\frac{\epsilon\omega_1}{2} t\right)\\ + \frac{1}{2}\left[X_i^{(1)} c_1 + X_i^{(2)} c_2\right] cos\,(\omega_1 t)\, cos\left(\frac{\epsilon\omega_1}{2} t\right)\end{aligned}$$

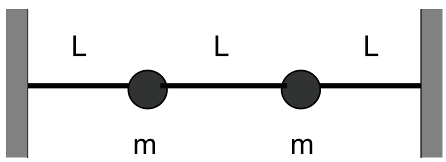
The first term on the right hand side of the above equation is represented by a low-frequency "amplitude modulation" $sin\left(\frac{\epsilon\omega_1}{2} t\right)$ on the response $sin\,(\omega_1 t)$. This amplitude-modulated response is shown in the following figure. As we see from this, this component corresponds to a "beating" response contribution, with the beat frequency given by $\frac{\epsilon\omega_1}{2}$ and a beat period of $\frac{4\pi}{\epsilon\omega_1}$. See the following plot.

The second term represents the same qualitative response of an amplitude-modulated motion with a beat frequency of $\frac{\epsilon\omega_1}{2}$.

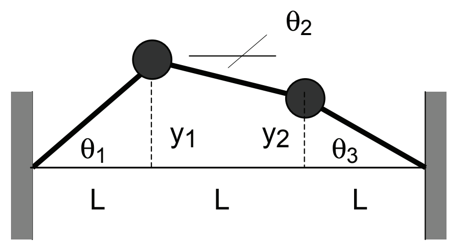
In summary, when a pair of frequencies are nearly equal, the modal response for this pair of modes with be a response at the common natural frequency modulated in amplitude by a sinusoid at half the frequency of the difference between the natural frequencies of this pair.

Example II.2.1

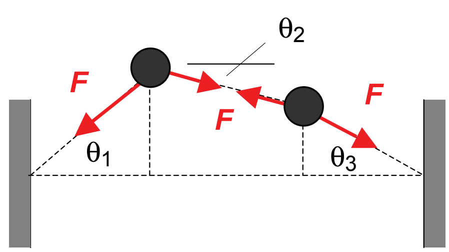
Two identical particles (each of mass m) are attached to a taut string (having a tension F and of negligible mass). The particles are allowed to move only in the horizontal plane shown. Assume that the initial tension F is large enough that the subsequent motion of the system will NOT affect the string tension. If the system is initially at rest with $y_1(0) = y_0$ and $y_2(0) = 0$, find the motion of the system upon release.



undeformed string



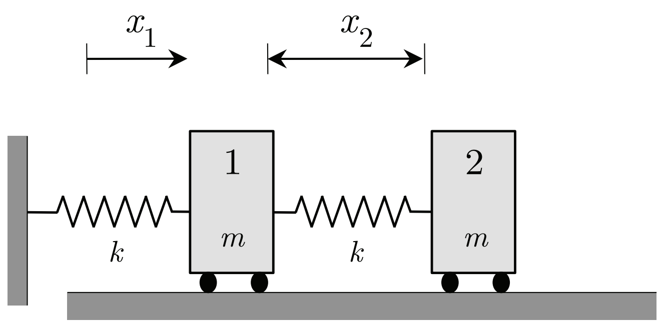
deformed string



FBD's of particles

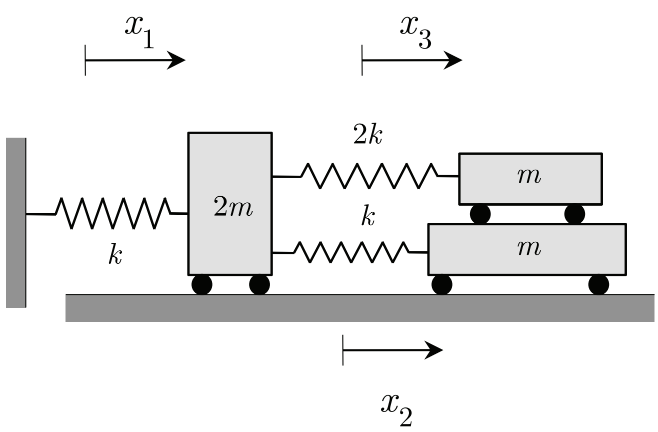
Example II.2.2

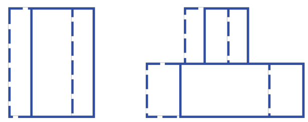
Find the natural frequencies and modal vectors for the two-DOF system shown using the absolute coordinate x_1 and relative coordinate x_2.



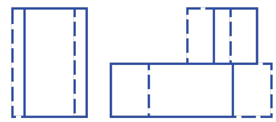
Example II.2.3

Find the natural frequencies and modal vectors for the three-DOF system shown using the absolute coordinates x_1, x_2 and x_3.

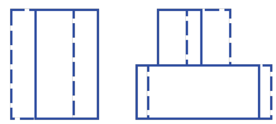




first mode, $\underline{X}^{(1)}$



second mode, $\underline{X}^{(2)}$



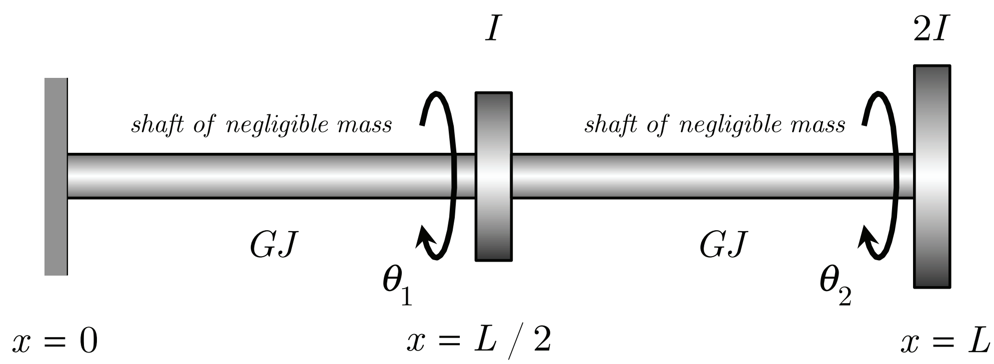
third mode, $\underline{X}^{(3)}$

Listing of Matlab program for finding natural frequencies and modal vectors:

```
%matlab program for Example II.6

clear
%solve via characteristic equation
c=[-2,10,-11,2]'; omega=sort(sqrt(roots(c)))

%solve using eigenvalue solver
clear omega
M=[2,0,0;0,1,0;0,0,1];                %define mass matrix
K=[4,-1,-2;-1,1,0;-2,0,2];            %define stiffness matrix
[X,d]=eig(K,M);                       %solve for modal vectors and evalues
[omega,id]=sort(sqrt(diag(d)));       %sort natural frequencies and modes
X=X(:,id);
```

Example II.2.4

Find the natural frequencies and modal vectors for the two-DOF system shown using the absolute coordinates θ_1 and θ_2.

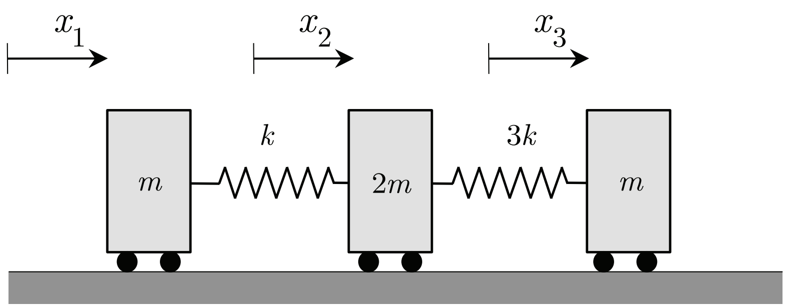


Matlab Code for Finding the Response Coefficients

```
clear

%mass and stiffness matrices
M=[1,0;0,2];
K=[2,-1;-1,1];

%initial conditions
x0=[1;1];
v0=[0;0];

%natural frequencies and modal vectors
[v,d]=eig(K,M);
[omega,id]=sort(sqrt(diag(d)));
v=v(:,id);

%normalize modal vectors
alpha=sqrt(diag(v'*M*v))';
v=v./[ones(2,1)*alpha];

%find response coefficients
c=v'*M*x0;
s=(v'*M*v0)./omega;
```

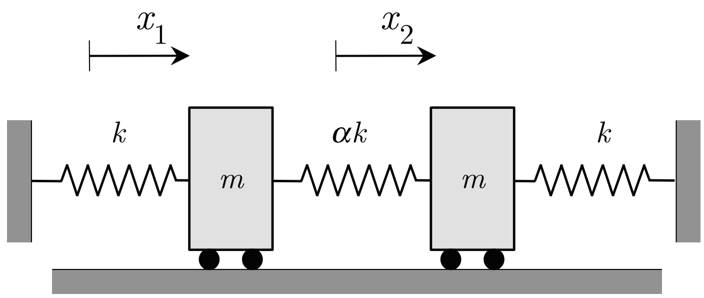
Example II.2.5

Consider the system shown below. (a) Verify that the system possess at least one rigid body mode; (b) Find the remaining natural frequencies and all mode shapes; and (c) Find the response to the initial conditions of $\vec{x}(0) = \vec{0}$, $\dot{x}_1(0) = 1$ and $\dot{x}_2(0) = \dot{x}_3(0) = 0$.



Example II.2.6

If $\alpha << 1$, find and describe the motion of the system if it starts from rest with $x_1(0) = A$ and $x_2(0) = 0$.



II.3 Free response: damped multi-DOF systems

In this section of the notes, we will study the undamped free response of multi-DOF systems governed by the following set of linear, 2nd order differential equations:

$$[M]\,\ddot{\vec{x}} + [C]\,\dot{\vec{x}} + [K]\,\vec{x} = \vec{0}$$

As seen in the Introduction to this chapter, the assumed solution form $\vec{x}(t) = \vec{X}e^{\lambda t}$ produces the following eigenvalue problem:

$$\left[\lambda^2\,[M] + \lambda\,[C] + [K]\right]\vec{X} = \vec{0}$$

Note that $\vec{X} = \vec{0}$ satisfies the above equation. However, $\vec{X} = \vec{0}$ implies that $\vec{x}(t) = 0$. This corresponds to no motion of the system (a "trivial" solution) and is not of interest to us. For $\vec{X} \neq 0$, the matrix $\left[\lambda^2\,[M] + \lambda\,[C] + [K]\right]$ must be singular:

$$det\left[\lambda^2\,[M] + \lambda\,[C] + [K]\right] = 0$$

Expanding the above determinant produces an $2N$th-order polynomial in λ:

$$b_2N\lambda^{2N} + b_{2N-1}\lambda^{2N-1} + b_{2N-2}\lambda^{2N-2}... + b_1\lambda + b_0 = 0$$

where the coefficients $b_0, b_1, ..., b_{2N}$ depend on the elements of $[M]$, $[C]$ and $[K]$. The above polynomial is known as the "characteristic equation" for the system. The characteristic equation will have $2N$ roots for λ (known as the "eigenvalues" of the system).

For each root λ_j $(j = 1, 2, ..., 2N)$ of the characteristic equation, we can solve for the corresponding vector $\vec{X}^{(j)}$ $(j = 1, 2, ..., 2N)$ using the following equations:

$$\left[\lambda_j^2\,[M] + \lambda_j\,[C] + [K]\right]\vec{X}^{(j)} = \vec{0}$$

This produces $2N$ vectors $\vec{X}^{(j)}$ $(j = 1, 2, ..., 2N)$ known as the "eigenvectors" for the system.

Discussion: comparison of the undamped and damped eigenvalue problems

Recall from before the following observations of the UNDAMPED eigenvalue problem:

- The characteristic polynomial contains only even powers of λ (that is, the characteristic polynomial for the undamped case is Nth order in λ^2).
- The roots for λ^2 are real and negative. Consequently, the roots for λ are purely imaginary: $\lambda_j = \pm i\omega_j$.
- The moduli of the purely imaginary roots for λ are the natural frequencies of the system.
- Since the roots for λ^2 are real, the eigenvectors vectors $\vec{X}$ are real.

Here, for the DAMPED eigenvalue problem:

- The characteristic polynomial contains both even and odd powers of λ. In general, we should expect the roots λ_j to be complex (containing both real and imaginary parts):

  $$\lambda_j = \nu_j + i\omega_j$$

- When complex roots λ_j exist, they will appear in complex conjugate pairs. That is, if $\lambda_j = \nu_j + i\omega_j$ is a root of the characteristic equation, then $\lambda_j = \nu_j - i\omega_j$ will also be a root.
- The real parts of the roots for λ_j are expected to be non-positive; that is, $\nu_j \leq 0$.
- When the roots for λ_j are complex, the eigenvectors $\vec{X}^{(j)}$ are will be complex. We will write the complex eigenvectors as:

  $$\vec{X}^{(j)} = \vec{u}^{(j)} + i\vec{v}^{(j)}$$

- As with the roots of the characteristic equation, complex eigenvectors will appear in complex conjugate pairs. That is, if $\vec{X}^{(j)} = \vec{u}^{(j)} + i\vec{v}^{(j)}$ is an eigenvector, then $\vec{X}^{(j)} = \vec{u}^{(j)} - i\vec{v}^{(j)}$ is also an eigenvector.

Free response of damped multi-DOF discrete systems

Let's assume that all roots of the characteristic equation are complex, and therefore, as discussed above, appear in complex conjugate pairs ($j = 1, 2, ..., N$):

$$\lambda_j = \nu_j \pm i\omega_j$$

As a result, the eigenvectors are also assumed to be complex, and appearing in complex conjugate pairs ($j = 1, 2, ..., N$):

$$\vec{X}^{(j)} = \vec{u}^{(j)} \pm i\vec{v}^{(j)}$$

If the eigenvectors $\vec{X}^{(j)}$ ($j = 1, 2, ..., N$) form an independent set, the solutions $\vec{X}^{(j)} e^{\lambda_j t}$ ($j = 1, 2, ..., 2N$) are independent. From this, we can say that the total free response is a linear combination of these solutions:

$$\begin{aligned}
\vec{x}(t) &= \sum_{j=1}^{N} \left[a_j e^{(\nu_j + i\omega_j)t} \left(\vec{u}^{(j)} + i\vec{v}^{(j)} \right) + b_j e^{(\nu_j - i\omega_j)t} \left(\vec{u}^{(j)} - i\vec{v}^{(j)} \right) \right] \\
&= \sum_{j=1}^{N} e^{\nu_j t} \left[a_j e^{i\omega_j t} \left(\vec{u}^{(j)} + i\vec{v}^{(j)} \right) + b_j e^{-i\omega_j t} \left(\vec{u}^{(j)} - i\vec{v}^{(j)} \right) \right] \\
&= \sum_{j=1}^{N} e^{\nu_j t} \left[a_j \left(cos\omega_j t + sin\omega_j t \right) \left(\vec{u}^{(j)} + i\vec{v}^{(j)} \right) + b_j \left(cos\omega_j t - sin\omega_j t \right) \left(\vec{u}^{(j)} - i\vec{v}^{(j)} \right) \right] \\
&= \sum_{j=1}^{N} e^{\nu_j t} \left[\left(a_j + b_j \right) \left(\vec{u}^{(j)} cos\omega_j t - \vec{v}^{(j)} sin\omega_j t \right) + i \left(a_j - b_j \right) \left(\vec{u}^{(j)} sin\omega_j t + \vec{v}^{(j)} cos\omega_j t \right) \right] \\
&= \sum_{j=1}^{N} e^{\nu_j t} \left[c_j \left(\vec{u}^{(j)} cos\omega_j t - \vec{v}^{(j)} sin\omega_j t \right) + s_j \left(\vec{u}^{(j)} sin\omega_j t + \vec{v}^{(j)} cos\omega_j t \right) \right] \\
&= \sum_{j=1}^{N} e^{\nu_j t} \left[\left(c_j \vec{u}^{(j)} + s_j \vec{v}^{(j)} \right) cos\omega_j t + \left(-c_j \vec{v}^{(j)} + s_j \vec{u}^{(j)} \right) sin\omega_j t \right]
\end{aligned}$$

where $c_j = a_j + b_j$ and $s_j = i \left(a_j - b_j \right)$.

Summary: free response of damped multi-DOF discrete systems

We have seen that the free response of a damped multi-DOF system is given by:

$$\vec{x}(t) = \sum_{j=1}^{N} e^{\nu_j t} \left[\left(c_j \vec{u}^{(j)} + s_j \vec{v}^{(j)} \right) cos\omega_j t + \left(-c_j \vec{v}^{(j)} + s_j \vec{u}^{(j)} \right) sin\omega_j t \right]$$

- The eigenvalues for damped systems are typically complex, with $\lambda_j = \nu_j + i\omega_j$. As seen above, the imaginary part ω_j is the frequency of the oscillatory portion of the response. The real part ν_j governs the rate of decay of the oscillations.

- The real and imaginary parts of the eigenvectors and play a more complicated role in the free response.

- The response above shows that the total free response is made up of modal contributions $e^{\nu_j t} \left[\left(c_j \vec{u}^{(j)} + s_j \vec{v}^{(j)} \right) cos\omega_j t + \left(-c_j \vec{v}^{(j)} + s_j \vec{u}^{(j)} \right) sin\omega_j t \right]$ might look similar to that for undamped systems but are quite different.

  - Here the modal contributions are decaying harmonics.

  - The shape of the modal contributions involves a combination of the real and imaginary parts of the eigenvectors. Recall that for the undamped problem, the shape of the response is simply given by the shape of the corresponding modal vector.

  - For undamped response, the modal contributions were synchronous (shape of response remains constant with time). Here for damped response, the shape of the modal contributions to the response changes with time (this might require some study of these modal contributions to make this observation). Therefore, the concept of a "mode shape" is not relevant for damped systems.

  - Note that for undamped response, $\nu_j = 0$ and $\vec{v}^{(j)} = \vec{0}$. For this case, the equation above for the damped response reduces to the undamped response, as expected.

- The undamped and damped responses for multi-DOF systems are summarized in the following table.

	Undamped	***Damped***
EOM's	$[M]\,\ddot{\underline{x}}+[K]\,\underline{x}=\underline{0}$	$[M]\,\ddot{\underline{x}}+[C]\,\dot{\underline{x}}+[K]\,\underline{x}=\underline{0}$
Solution form	$\underline{x}(t)=\underline{X}\,e^{\lambda t}$	$\underline{x}(t)=\underline{X}\,e^{\lambda t}$
Eigenvalue problem	$\left[\lambda^2[M]+[K]\right]\underline{X}=\underline{0}$	$\left[\lambda^2[M]+\lambda[C]+[K]\right]\underline{X}=\underline{0}$
Char. eqn.	$a_N\lambda^{2N}+a_{N-1}\lambda^{2(N-1)}+\ldots$ $+a_1\lambda^2+a_0=0$ (only *even* powers of λ)	$a_{2N}\lambda^{2N}+a_{2N-1}\lambda^{2N-1}+\ldots$ $+a_2\lambda^2+a_1\lambda+a_0=0$ (both even AND odd powers of λ)
Roots of char. eqn.	Purely *imaginary* (complex conjugate pairs): $\lambda_j=\pm\, i\omega_j$ (harmonic response)	Generally *complex* (complex conjugate pairs): $\lambda_j=\nu_j\pm i\omega_j$ (decaying harmonic response)
Eigenvectors, $\underline{X}^{(j)}$	Real (modal vectors) and are *orthogonal* through [M] and [K].	Generally complex (complex conjugate pairs): $\underline{X}^{(j)}=\underline{u}^{(j)}\pm i\underline{v}^{(j)}$ and do NOT possess the orthogonality properties of the undamped modes.
Solution	$\underline{x}(t)=\sum_{j=1}^{N}\left[c_j\cos\omega_j t+s_j\sin\omega_j t\right]\underline{X}^{(j)}$	$\underline{x}(t)=\sum_{j=1}^{N}e^{\nu_j t}\Big[\left(c_j\underline{u}^{(j)}+s_j\underline{v}^{(j)}\right)\cos\omega_j t$ $+\left(-c_j\underline{v}^{(j)}+s_j\underline{u}^{(j)}\right)\sin\omega_j t\Big]$

Special topic: Proportional damping

Up to this point, we have considered a general form of the damping matrix $[C]$. Here we will investigate a special form of the damping matrix that is a linear combination of the symmetric mass and stiffness matrices of the system:

$$[C] = \alpha[M] + \beta[K]$$

where α and β are scalar constants. At first glance, this might appear to be a rather artificial description of damping. However, there are many systems for which this description closely resembles the actual damping. For example, $[C] = \alpha[M]$ (mass proportional or external damping) would arise in a system where dashpots are connected between ground and each particle in the system, with each damping coefficient being proportional to the particle mass. $[C] = \beta[K]$ (stiffness proportional or internal damping) would arise in a system where dashpots exist at all locations where springs exist, with each damping coefficient being proportional to the corresponding spring stiffness. The general form shown above $[C] = \alpha[M] + \beta[K]$ is known as "proportional" (or, Rayleigh) damping. Rayleigh damping is also used when the actual damping mechanism is not well known; in that case, this model can be a reasonable approximation to the actual damping.

With this form of damping, the free response EOM's become:

$$[M]\,\ddot{\vec{x}} + [\alpha[M] + \beta[K]]\dot{\vec{x}} + [K]\,\vec{x} = \vec{0}$$

Here we will consider the following coordinate transformation:

$$\vec{x}(t) = \sum_{j=1}^{N} \vec{X}^{(j)} p_j(t)$$

where $\vec{X}^{(j)}$ are the UNDAMPED modal vectors for the problem and $p_j(t)$ are a set of to-be determined "modal response coefficients".

Substitute the above coordinate transformation into the above Rayleigh-damped EOM's and premultiply by the transpose of one of the undamped modal vectors $\vec{X}^{(k)}$:

$$\sum_{j=1}^{N} \vec{X}^{(k)T}\,[M]\,\vec{X}^{(j)}\ddot{p}_j + \sum_{j=1}^{N} \vec{X}^{(k)T}\,(\alpha\,[M] + \beta[K])\,\vec{X}^{(j)} p_j + \sum_{j=1}^{N} \vec{X}^{(k)T}[K]\vec{X}^{(j)}\dot{p}_j = \vec{0}$$

$$\sum_{j=1}^{N} \vec{X}^{(k)T}\,[M]\,\vec{X}^{(j)}\,(\ddot{p}_j + \alpha\dot{p}_j) + \sum_{j=1}^{N} \vec{X}^{(k)T}[K]\vec{X}^{(j)}\,(\beta\dot{p}_j + p_j) = \vec{0}$$

Recall that the undamped modal vectors are orthogonal: $\vec{X}^{(k)T}\,[M]\,\vec{X}^{(j)} = \vec{X}^{(k)T}\,[K]\,\vec{X}^{(j)} = 0$ for $j \neq k$. Furthermore, for mass-normalized modes, we have: $\hat{\vec{X}}^{(j)T}\,[M]\,\hat{\vec{X}}^{(j)} = 1$ and $\hat{\vec{X}}^{(j)T}\,[K]\,\hat{\vec{X}}^{(j)} = \omega_j^2$, where ω_j are the undamped natural frequencies. With these observations, the summations in the above are eliminated and the above reduce to:

$$\ddot{p}_j + \left(\alpha + \omega_j^2\beta\right)\dot{p}_j + \omega_j^2 p_j = 0$$

We see the the Rayleigh-damped modal EOM's are uncoupled; that is, a single EOM contains only a single modal coordinate $p_j(t)$, and we have N single-DOF EOM's. Here we will put these uncoupled EOM's in the standard form for single-DOF systems:

$$\ddot{p}_j + 2\zeta_j\omega_j\dot{p}_j + \omega_j^2 p_j = 0$$

where $\zeta_j = (\alpha/\omega_j + \beta\omega_j)/2$ is the damping ratio for the pth modal coordinate.

We now solve these uncoupled EOM's as we did for the free response of single-DOF systems. Suppose that all modes are "underdamped"; that is, $\zeta_j < 1$ for all $j = 1, 2, ..., N$. For this, the solutions for the free response modal equations are:

$$p_j(t) = e^{-\zeta_j\omega_j t}\left(c_j cos\omega_{dj}t + s_j sin\omega_{dj}t\right)$$

where $\omega_{dj} = \omega_j\sqrt{1-\zeta_j^2}$.

Substituting these modal responses back into the original coordinate transformation above gives:

$$\vec{x}(t) = \sum_{j=1}^{N}\vec{X}^{(j)}p_j(t) = \sum_{j=1}^{N}\vec{X}^{(j)}e^{-\zeta_j\omega_j t}\left(c_j cos\omega_{dj}t + s_j sin\omega_{dj}t\right)$$

From this, we see that the special Rayleigh-damped systems share the same type of solution form as the corresponding undamped systems: the modal contributions correspond to synchronous motion. However, here the modal contributions are exponentially-decaying harmonic responses. Also from this, we see that each mode has a damping ratio ζ_j and damped natural frequency ω_{dj} assigned to it. These parameters are directly related back to the undamped natural frequencies ω_j and the Rayleigh damping parameters α and β.

Example II.3.1
Find the eigenvalues and eigenvectors for the two-DOF system shown. Compare your results with those of the undamped system. Find the response corresponding to the system being released from rest with $x_1(0) = x_2(0) = 1$.

Is this system proportionally damped? If not, what modifications can be made to the model to make it proportionally damped?