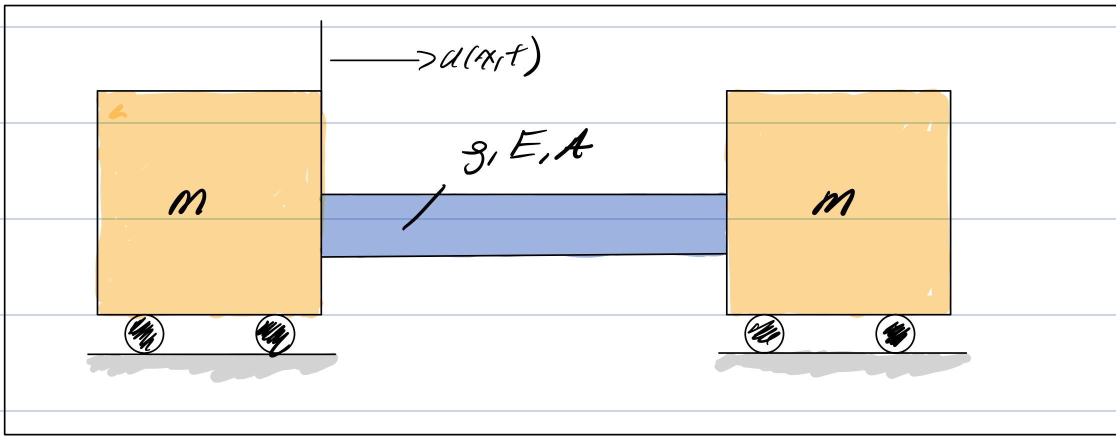
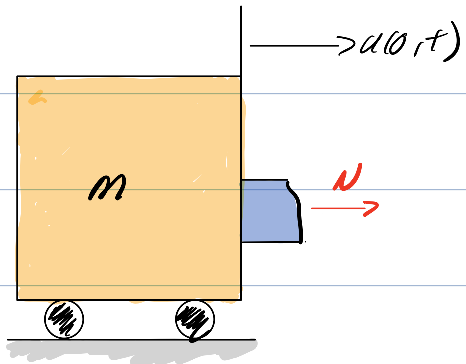
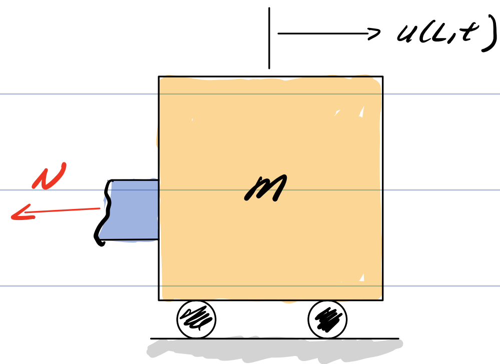
Homework 4.2







$\overset{+}{\rightarrow} \Sigma F: N = m\ddot{u}(0,t)$

① $EA\dfrac{\partial u(0,t)}{\partial x} = m\ddot{u}(0,t)$

$\overset{+}{\rightarrow} \Sigma F: -N = m\ddot{u}(L,t)$

② $-EA\dfrac{\partial u(L,t)}{\partial x} = m\ddot{u}(L,t)$

The boundary conditions

① $EA\dfrac{\partial u}{\partial x}(0,t) = m\ddot{u}(0,t)$

② $-EA\dfrac{\partial u}{\partial x}(L,t) = m\ddot{u}(L,t)$

The governing equation: $EA\dfrac{\partial u}{\partial x} = \rho A\dfrac{\partial^2 u}{\partial t^2}$

Separable soln. $u(x,t) = U(x)\,T(t)$

$$E\,U''(x)\,T(t) = \rho\,U(x)\,\ddot{T}(t) \longrightarrow \frac{E\,U''(x)}{\rho\,U(x)} = \frac{\ddot{T}(t)}{T(t)} = -\omega^2$$

① $$U''(x) + \frac{\omega^2 \rho}{E}\,U(x) = 0 \longrightarrow U''(x) + \beta U(x) = 0$$

$$\beta = \omega\sqrt{\rho/E}$$

② $$\ddot{T}(t) + \omega^2 T(t) = 0$$

The solutions are of the form:

$T(t) = C\cos\omega t + S\sin\omega t$, $U(x) = a\cos\beta x + b\sin\beta x$

Now evaluate boundary conditions

$U(x) = a\cos\beta x + b\sin\beta x$

$U'(x) = -a\beta\sin\beta x + b\beta\cos\beta x$

$U(0) = a$, $U(L) = a\cos\beta L + b\sin\beta$

$U'(0) = b\beta$, $U'(L) = \beta(-a\sin\beta L + b\cos\beta L)$

$$EA\frac{\partial u(0,t)}{\partial x} = m\frac{\partial^2 u(0,t)}{\partial t^2}$$

$$EA\,U'(0)T(t) = +m\,U(0)\ddot{T}(t) = -m\omega^2 U(0)T(t)$$

$$EA\,U'(0) = -m\omega^2 U(0)$$

$$EA\frac{\partial u(L,t)}{\partial x} = -m\frac{\partial^2 u(L,t)}{\partial t^2}$$

$$EA\,U'(L)T(t) = -m\,U(L)\ddot{T}(t) = m\omega^2 U(L)T(t)$$

$$EA\,U'(L) = m\omega^2 U(L)$$

$$EA\,U'(0) = -m\omega^2 U(0)$$

$$EA\,a\beta = -m\omega^2 b \qquad \omega = \beta\sqrt{E/\rho}$$

$$EA\,a\beta = -m\beta^2 \frac{E}{\rho}\, b$$

$$A\,b\,\beta = \frac{-m\beta^2 a}{\rho}$$

$$b\beta = \frac{-m\beta^2 a}{\rho A} \rightarrow b = \frac{-m\beta}{\rho A}a$$

$$b = \frac{-m\beta L}{\rho A L}a$$

$$\underline{b = -\beta L a}$$

$$EAu'(L) = m\omega^2 u(L)$$

$$u(L) = a\cos\beta L + b\sin\beta$$

$$u'(L) = \beta(-a\sin\beta L + b\cos\beta L)$$

$$EA\beta(-a\sin\beta L + b\cos\beta L) = m\omega^2(a\cos\beta L + b\sin\beta L)$$

$$EA\beta(-a\sin\beta L - \beta L a\cos\beta L) = m\omega^2(a\cos\beta L - \beta L a\sin\beta L)$$

$$\frac{EA\beta}{m\omega^2}(\sin\beta L + \beta L\cos\beta L) = (-\cos\beta L + \beta L\sin\beta L)$$

$$\hookrightarrow \quad \frac{EA\beta}{m\omega^2} = \frac{EA\beta}{m\beta^2 E/g} = \frac{gA\beta}{m\beta^2} = \frac{gAL}{m\beta L} = 1/\beta L$$

$$(\sin\beta L + \beta L\cos\beta L) = \beta L(-\cos\beta L + \beta L\sin\beta L)$$

$$2\beta L\cos\beta L = ((\beta L)^2 - 1)\sin\beta$$

$$\tan\beta L = \frac{2\beta L}{(\beta L)^2 - 1} \sim CE$$

Note there are multiple ways to write this - all give same roots.

Rigid body Motion

One root is $\beta L = 0 \longrightarrow \beta = 0, \ \omega = 0$

The BVP $\longrightarrow u''(x) + \cancel{\frac{\omega^2 E}{g}} u(x) = 0$

$u(x) = cx + d \quad , \quad u'(x) = c$

What are c and d

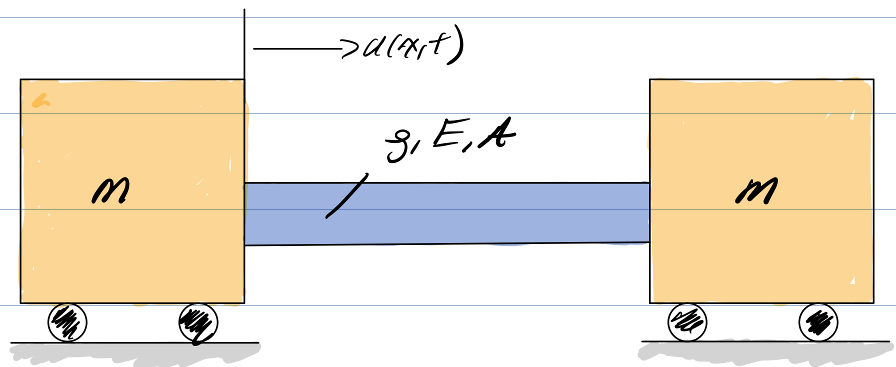
$EAu'(0) = -m\overset{\to 0}{\cancel{\omega^2}}u(0) = 0$

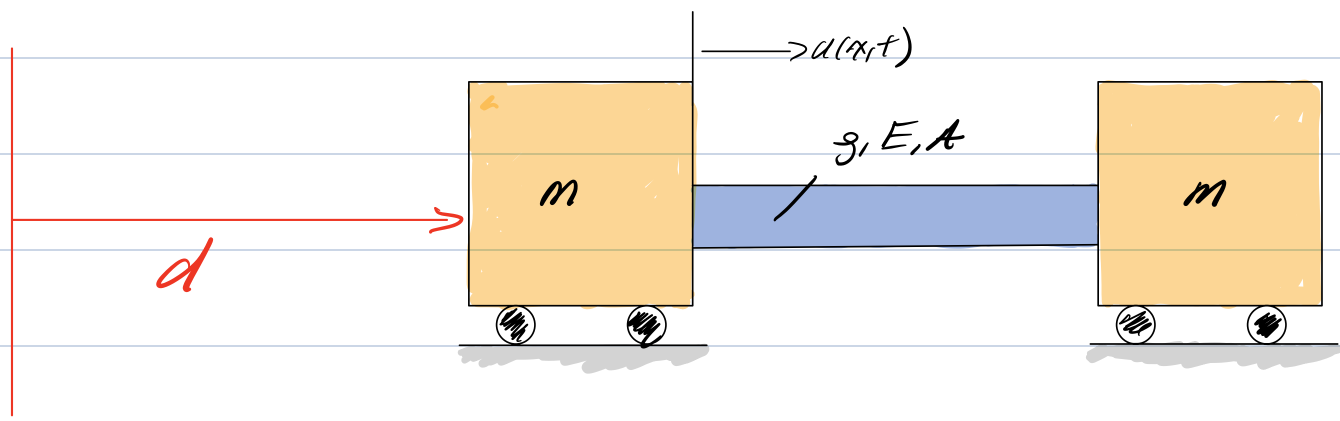
$EAu'(L) = m\overset{\to 0}{\cancel{\omega^2}}u(L) = 0$

$\therefore \; u'(0) = 0 = c \qquad c = 0$

$\quad u'(L) = 0 = cL$

$u(x) = d \rightarrow$ is a constant motion





Rigid Body translation

Vibration

Again examine

$$\tan \beta L = \frac{2\beta L}{(\beta L)^2 - 1}$$

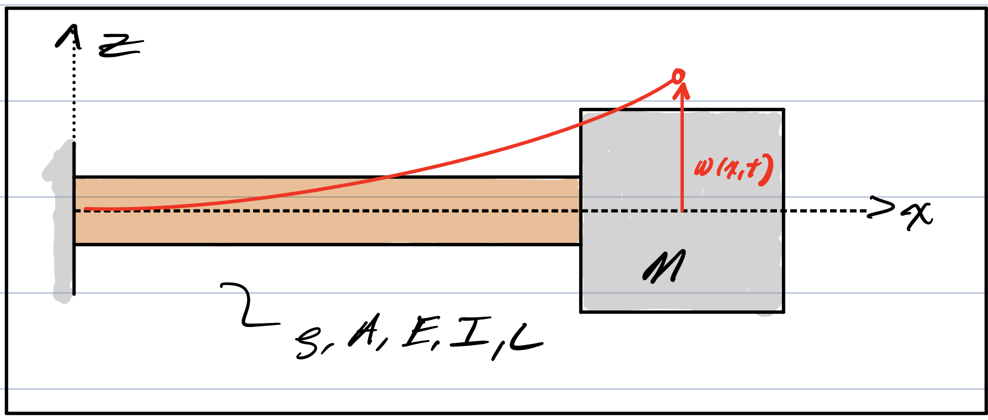
From Matlab code

$\beta_1 L = 1.3065 \longrightarrow \omega_1 = 1.3065 \sqrt{E/gL^2}$

$\beta_2 L = 3.6732 \longrightarrow \omega_2 = 3.6732 \sqrt{E/gL^2}$

$\beta_3 L = 6.5846 \longrightarrow \omega_3 = 6.5846 \sqrt{E/gL^2}$

Homework 4.3



1^{st} write the EOM

$$EI\frac{\partial^4 w}{\partial x^4} = -\rho A\frac{\partial^2 w}{\partial t^2}$$

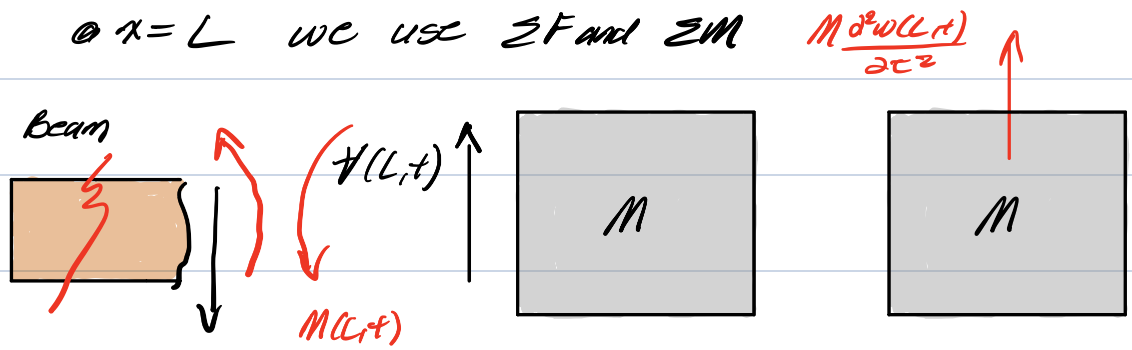
The boundary conditions can be written as

@ $x=0$

① $w(0,t) = W(0)\,T(t) = 0 \longrightarrow \boxed{W(0)=0}$

② $\frac{\partial w}{\partial x}(0,t) = W'(0)\,T(t) = 0 \longrightarrow \boxed{W'(0)=0}$

@ $x=L$ we use ΣF and ΣM



$\circlearrowleft\ \Sigma M: \quad M(L,t) = EI\frac{\partial^2 w(L,t)}{\partial x^2} = 0$

$+\uparrow\ \Sigma F_x: \quad +V(L,t) = M\frac{\partial^2 w(L,t)}{\partial t^2}$

③ $EI\dfrac{\partial^2 w(L,t)}{\partial x^2} = EI\, W''(L)T(t) = 0 \rightarrow \boxed{W''(L) = 0}$

④ $EI\dfrac{\partial^3 w(L,t)}{\partial x^3} = M\dfrac{\partial^2 w(L,t)}{\partial t^2}$

$$EI\, W'''(L)T(t) = M\, W(L)\overset{-w^2T(t)}{\ddot{T}(t)}$$

$$EI\, W'''(L) = -M\, W(L)\, w^2$$

$$\boxed{W'''(L) = -w^2 \frac{M}{EI} W(L)}$$

The spatial solution is

$W(x) = a\cosh\beta x + b\sinh\beta x + c\cos\beta x + d\sin\beta x$

where $\beta^2 = \sqrt{\rho A/EI}\; w$

① $W(0) = a\overset{1}{\cosh 0} + b\overset{0}{\sinh 0} + c\overset{1}{\cos 0} + d\overset{0}{\sin 0} = 0$

② $W'(0) = \beta(a\overset{0}{\sinh 0} + b\overset{1}{\cosh 0} - c\overset{0}{\sin 0} + d\overset{1}{\cos 0}) = 0$

① $0 = a + c \rightarrow c = -a$

② $0 = \beta(b + d) \rightarrow d = -b \qquad \beta \neq 0$

Now,

$W(x) = a(\cosh\beta x - \cos\beta x) + b(\sinh\beta x - \sin\beta x)$

$W''(x) = \beta^2(a\cosh\beta x + a\cos\beta x + b\sinh\beta x + b\sin\beta x)$

$W'''(x) = \beta^3(a\sinh\beta x - a\sin\beta x + b\cosh\beta x + b\cos\beta x)$

③ $0 = \beta^2 (a \cosh \beta L + a \cos \beta L + b \sinh \beta L + b \sin \beta L)$

$$0 = a(\cosh \beta L + \cos \beta L) + b(\sinh \beta L + \sin \beta L)$$

④ $0 = \beta^3 (a \sinh \beta L - a \sin \beta L + b \cosh \beta L + b \cos \beta L)$
$$+ \frac{\omega^2 M}{EI} (+a \cosh \beta L - a \cos \beta L + b \sinh \beta L - b \sin \beta L)$$

$$\omega = \sqrt{\frac{EI}{\rho A}}\, \beta^2 \quad \text{and} \quad \omega^2 = \frac{EI}{\rho A} \beta^4$$

$$\frac{\omega^2 M}{EI} = \frac{EI}{\rho A} \frac{M}{EI} \beta^4 = \frac{M}{\rho A} \beta^4$$

④ $0 = \beta^3 (a \sinh \beta L - a \sin \beta L + b \cosh \beta L + b \cos \beta L)$
$$+ \frac{M}{\rho A} \beta^4 (a \cosh \beta L - a \cos \beta L + b \sinh \beta L - b \sin \beta L)$$

$$0 = (a \sinh \beta L - a \sin \beta L + b \cosh \beta L + b \cos \beta L)$$
$$+ \frac{M}{\rho A} \beta (a \cosh \beta L - a \cos \beta L + b \sinh \beta L - b \sin \beta L)$$

multiply $\frac{M}{\rho A} \beta$ by L/L

$$0 = (a \sinh \beta L - a \sin \beta L + b \cosh \beta L + b \cos \beta L)$$
$$+ \frac{M}{\cancel{\rho A L}} \beta L (a \cosh \beta L - a \cos \beta L + b \sinh \beta L - b \sin \beta L)$$

Put in matrix form

$$\underbrace{\begin{bmatrix} (\cosh\beta L + \cos\beta L) & (\sinh\beta L + \sin\beta L) \\ \sinh\beta L - \sin\beta L & \cosh\beta L + \cos\beta L \\ + \beta L(\cosh\beta L - \cos\beta L) & + \beta L(\sinh\beta L - \sin\beta L) \end{bmatrix}}_{D} \begin{Bmatrix} a \\ b \end{Bmatrix} = 0$$

Simplified these calculations in the Matlab code

$CE = \det(D)$

$$\boxed{CE = 1 + \cos\beta L \cosh\beta L + \beta L \cos\beta L \sinh\beta L - \beta L \cosh\beta L \sinh\beta L}$$

From Matlab Code ...

βL_1	1.2479
βL_2	4.0311
βL_3	7.1341
βL_4	10.2566
βL_5	13.3878

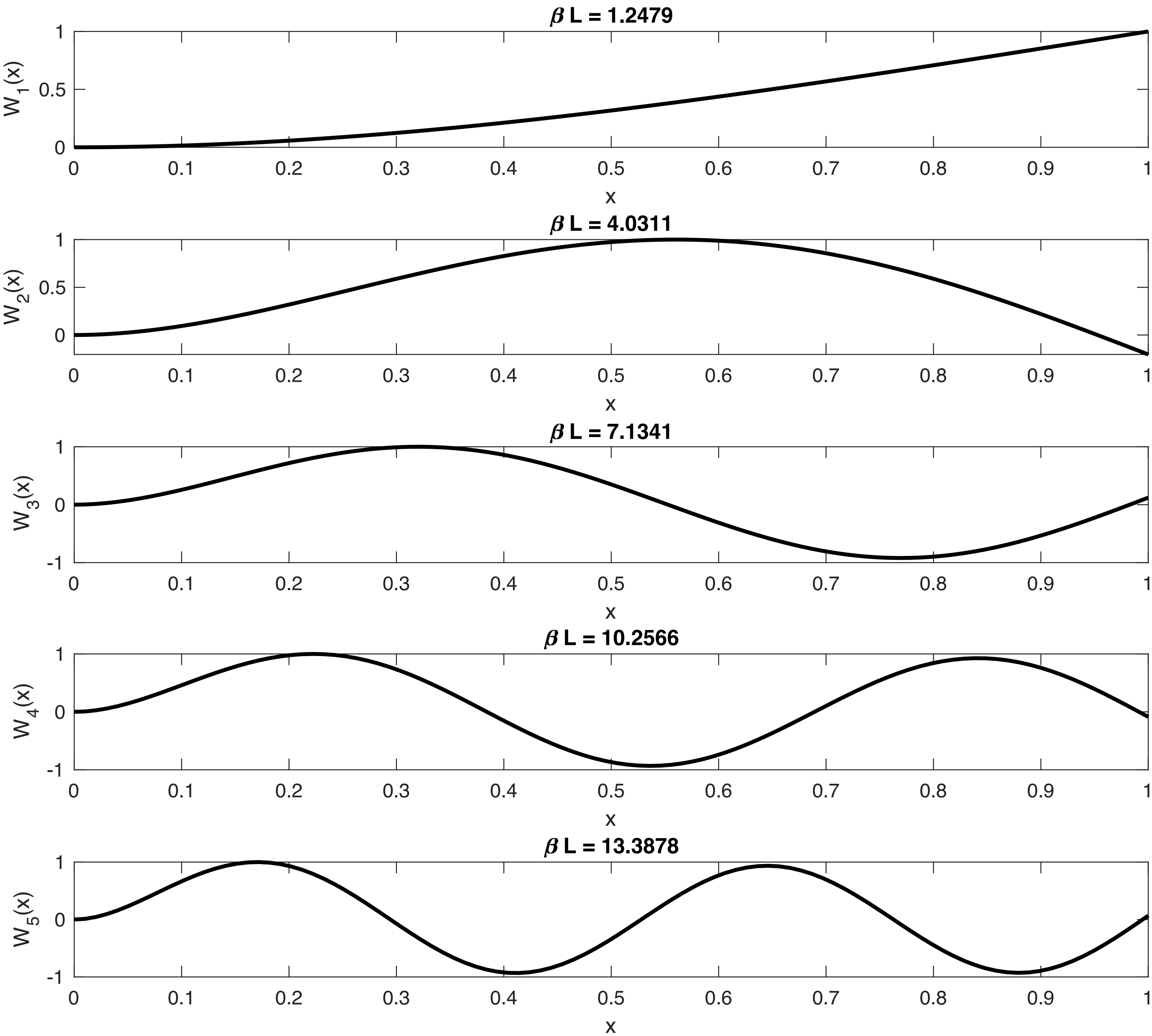
Now, solve for Modeshapes using 1st row of matrix equation

$$(\cosh \beta L - \cos \beta L)\,a + (\sinh \beta L - \sin \beta L)\,b = 0$$

$$b = \frac{(\cosh \beta L - \cos \beta L)}{(\sinh \beta L - \sin \beta L)}\,a$$

$$W(x) = a(\cosh \beta x - \cos \beta x) + b(\sinh \beta x - \sin \beta x)$$

$$\boxed{W(x) = a(\cosh \beta x - \cos \beta x) + \frac{(\cosh \beta L - \cos \beta L]}{(\sinh \beta L - \sin \beta L}(\sinh \beta x - \sin \beta x)}$$

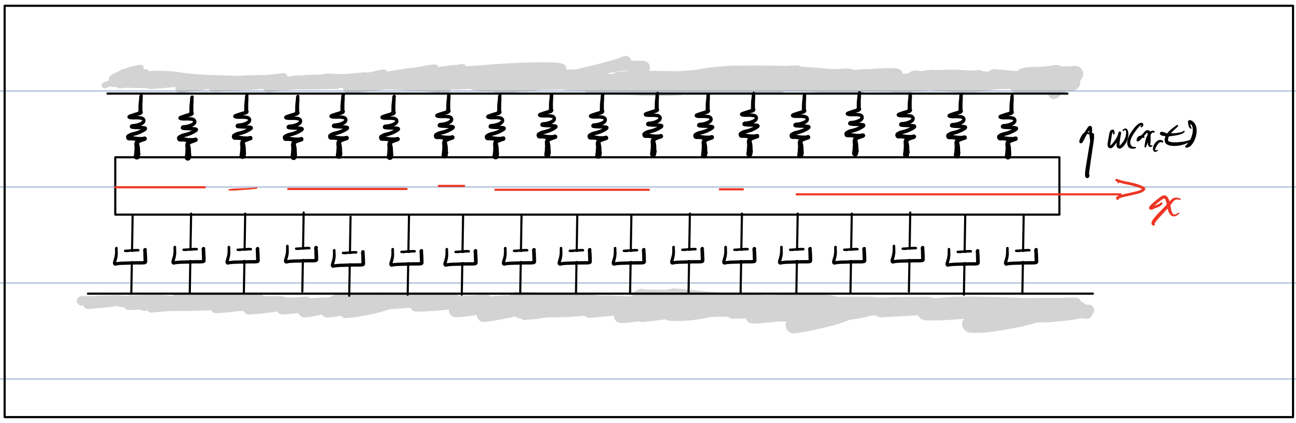


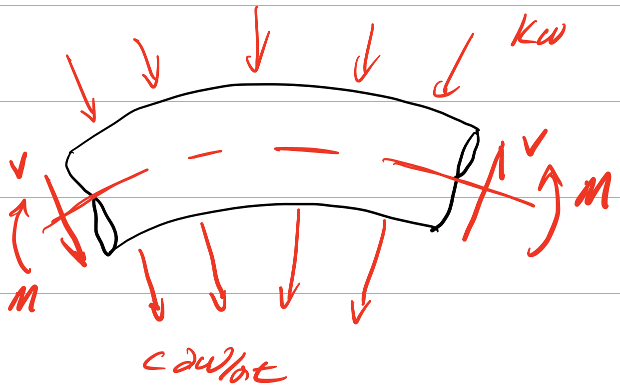
β L = 1.2479
W1(x)
β L = 4.0311
W2(x)
β L = 7.1341
W3(x)
β L = 10.2566
W4(x)
β L = 13.3878
W5(x)
x

```
clc
clear
close all
fprintf(['\n\n\nStarting file >>' mfilename '<< at ' datestr(now,0) '\n\n']);
format long
% HW  5.3

% x= Beta*L
syms B x l a b c d M E I w rho L A BL

W    = a*cosh(B*x)-a*cos(B*x)+b*sinh(B*x)-b*sin(B*x)
Wp   = diff(W,x)
Wpp  = diff(Wp,x)
Wppp = diff(Wpp,x)

% Evaluate at endpoint x =L
x   = L;
WL    = eval(W)
WppL  = eval(Wpp)
WpppL = eval(Wppp)

E1 = WppL
E2 = WpppL+w^2*L/L*M/(E*I)*WL

% Math Simplification - this is a guide you need to look at the problem on
% paper to realize how to use Matlab to simlify this....

E1 = E1;
E2 = subs(E2,w^2,(E*I)/(rho* A)*B^4);
E2 = expand(E2/B^3);

% get rid of B*L and replace with y
E1 = subs(E1,B*L,BL);
E2 = subs(E2,B*L,BL);

E2 = subs(E2/L,B,  BL);
E2 = subs(E2, {M, rho, A, L}, {1, 1, 1, 1});
E1 = simplify(E1/B^2);

% Write as matrix pull out coefficents
BC11 = coeffs(E1,a);
BC12 = coeffs(E1,b);
BC21 = coeffs(E2,a);
BC22 = coeffs(E2,b);

% order from lowest to highest power- lowest is coefficient^0,
%highest is coefficent^1;
BC11 = BC11(2);
BC12 = BC12(2);
BC21 = BC21(2);
BC22 = BC22(2);
BC   = [BC11 BC12; BC21 BC22];

CE  = simplify(det(BC));
BL  = linspace(0,5*pi,10^4);
CEv = eval(CE);

% This CE does not have  plot that we can easily identify our iniitial
```

```
% guess
% In order to get initial guess we look at places where the is a sign
% change in the CE
[Val1, loc1] = find(abs(diff(sign(CEv)))==2); % This time loc1 is indices
% Initial Guesses
x0v = BL(loc1);
options_all=[];
% set your length L
L=1;
for i =1:5
    x0 = x0v(i);
    fprintf(['\n\n\n >> Mode' num2str(i) '<<  \n\n']);
    % Fzero
    [xval, fval, exitflag] = fzero(@CEfunction,x0,options_all,CE);
    betaL(i) = xval;%This is Beta*L
    beta(i)  = betaL(i)/L; % THis is Beta

end
%Solve for Modeshapes
%solve fofr a constant
bsol=solve(E1==0,b)
%plug into W(x)
Wmode= simplify(subs(W,b,bsol));
pretty(Wmode)
%Plot mode shapes
x  =linspace(0,L,100);
figure
for i =1:5
    B  = beta(i);
    BL = betaL(i);
    a  = 1;
    WmodeP = eval(Wmode);
    Wmax   = max(abs(WmodeP));
    WmodeP = WmodeP/Wmax;
    subplot(5,1,i)
    line(x,WmodeP,'linewidth',2,'color','k')
    xlabel('x')
    ylabel(['W_',num2str(i),'(x)'])
    title(['\beta L = ', num2str(BL)])
    box on
end


function [F] = CEfunction(BL,CE)
% Pass the symbolic characteristic equation into a function
F =  eval(CE);
end
```

Homework 4.4





EOM: $EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = -kw - c\frac{\partial w}{\partial t}$

$$EI\frac{\partial^4 w}{\partial x^4} + c\frac{\partial w}{\partial t} + kw + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

Boundary Conditions: $EI\frac{\partial^2 w(0,t)}{\partial x^2} = 0$, $EI\frac{\partial^2 w(L,t)}{\partial x^2} = 0$

$$EI\frac{\partial^3 w(0,t)}{\partial x^3} = 0 , \quad EI\frac{\partial^3 w(L,t)}{\partial x^3} = 0$$

Rearrange governing equation

$$EI\frac{\partial^4 w}{\partial x^4} + c\frac{\partial w}{\partial t} + kw + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

$$EI\, W^4(x)\, T(t) + c\, W(x)\, \dot{T}(t) + k\, W(x)\, T(t) + \rho A\, W(x)\, \ddot{T}(t) = 0$$

$$EI\, W^4(x)\, T(t) + k\, W(x)\, T(t) = -\rho A\, W(x)\, \ddot{T}(t) - c\, W(x)\, \dot{T}(t)$$

divide by $\rho A W(x) T(t)$

$$-\left(\frac{EI}{\rho A}\frac{W^4(x)}{W(x)} + \frac{k}{\rho A}\right) = \frac{\ddot{T}(t)}{T(t)} + \frac{c}{\rho A}\frac{\dot{T}(t)}{T(t)} = -\omega^2$$

$$-\frac{EI}{\rho A}\frac{W^4(x)}{W(x)} - \frac{k}{\rho A} + \omega^2 = 0$$

$$W^4(x) + \left(\frac{-\rho A \omega^2}{EI} + \frac{k}{EI}\right) W(x) = 0$$

① $W^4(x) - \left(\frac{\rho A \omega^2}{EI} - \frac{k}{EI}\right) W(x) = 0$

② $\ddot{T}(t) + \frac{c}{\rho A}\dot{T}(t) + \omega^2 T(t) = 0$

Rename: $\beta^4 = \frac{\rho A \omega^2}{EI} - \frac{k}{EI}$

$$2\zeta\omega = c/\rho A$$

① $W^4 - \beta^4 W = 0$

② $\ddot{T} + 2\zeta\omega \dot{T} + \omega^2 T = 0$

Soln of ② $T(t) = e^{-\zeta\omega t}(C\cos\omega_d t + S\sin\omega_d t)$

$\omega_d = \omega\sqrt{1-\zeta^2}$

Not needed for correct answer

$$W(x) = a\cosh\beta x + b\sinh\beta x + c\cos\beta x + d\sin\beta x$$

$$W'(x) = a\beta\sinh\beta x + b\beta\cosh\beta x - c\beta\sin\beta x + d\beta\cos\beta x$$

$$W''(x) = a\beta^2\cosh\beta x + b\beta^2\sinh\beta x - c\beta^2\cos\beta x - d\beta^2\sin\beta x$$

$$W'''(x) = a\beta^3\sinh\beta x + b\beta^3\cosh\beta x + c\beta^3\sin\beta x - d\beta^3\cos\beta x$$

with Boundary conditions

$W''(0)=0,\ W'''(0)=0,\ W''(L)=0,\ W'''(L)=0$

$W''(0) = a\beta^2 - c\beta^2 = 0 \longrightarrow a=c$ $\underline{\text{if } \beta\neq 0}$

$W'''(0) = b\beta^3 - d\beta^3 = 0 \longrightarrow b=d$ $\underline{\text{if } \beta\neq 0}$

Now,

$$W(x) = a(\cosh\beta x + \cos\beta x) + b(\sinh\beta x + \sin\beta x)$$

$$W''(x) = a\beta^2(\cosh\beta x - \cos\beta x) + b\beta^2(\sinh\beta x - \sin\beta x)$$

$$W'''(x) = a\beta^3(\sinh\beta x + \sin\beta x) + b\beta^2(\cosh\beta x - \cos\beta x)$$

Evaluate at $x = L$

$$W''(L) = a\beta^2(\cosh\beta L - \cos\beta L) + b\beta^2(\sinh\beta L - \sin\beta L) = 0$$

$$a = \frac{-(\sinh\beta L - \sin\beta L)b}{(\cosh\beta L - \cos\beta L)}$$

Plug into last BC

$$W'''(L) = a\beta^3(\sinh\beta L + \sin\beta L) + b\beta^3(\cosh\beta L - \cos\beta L) = 0,$$

$$= \frac{-\beta^3(\sinh\beta L - \sin\beta L)(\sinh\beta L + \sin\beta L)\,b}{\cosh\beta L - \cos\beta L}$$

$$+ \frac{\beta^3(\cosh\beta L - \cos\beta L)(\cosh\beta L - \cos\beta L)\,b}{\cosh\beta L - \cos\beta L} = 0,$$

$$= \frac{\beta^3(-\sinh\beta L + \sin\beta L)(\sinh\beta L + \sin\beta L)\,b}{\cosh\beta L - \cos\beta L}$$

$$+ \frac{\beta^3(\cosh\beta L - \cos\beta L)(\cosh\beta L - \cos\beta L)}{\cosh\beta L - \cos\beta L} = 0,$$

$$= \frac{\beta^3(2 - 2\cosh\beta L\cos\beta L)b}{\cosh\beta L - \cos\beta L} = 0$$

Leads to characteristic equation: $1 = \cosh \beta L \cos \beta L$ ~ admits 2 zero roots

The roots are $\beta_1 L = 0$, $\beta_3 L = 4.73$, $\beta_4 L = 7.8532$

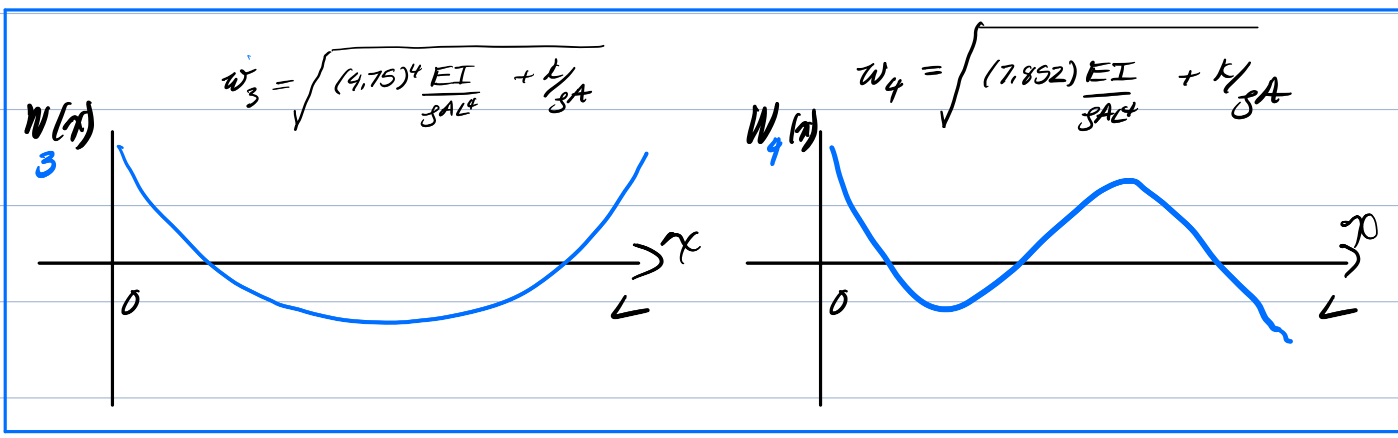
$\beta_1 L = 0$ has multiplicity of 2, ~ from matlab

The non zero mode shapes are given by

$$W(x) = \left(\frac{-b(\sinh \beta L - \sin \beta L)}{\cosh \beta L - \cos \beta L}\right)(\cosh \beta x + \cos \beta x) + b(\sinh \beta x + \sin \beta x)$$

$$\omega_3 = \sqrt{(4.75)^4 \frac{EI}{\rho A L^4} + k/\rho A}$$

$$\omega_4 = \sqrt{(7.8532)^4 \frac{EI}{\rho A L^4} + k/\rho A}$$



Flexural Modes

Note @ $\beta L = 0$

$$\omega_{1,2} = \sqrt{k/\rho A}$$

~ zero strain mode

behaves as a rigid beam

How do we find $\beta L = 0$ modes?

Go back to BVP

$$W''''(x) - \cancel{\beta^2}^{\,0} W(x) = 0$$

$W''''(x) = 0$ is the new governing mode shape equation

$W_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i \quad i = 0, 1$ since we have 2 RB modes

$W_0(x) = a_0 x^3 + b_0 x^2 + c_0 x + d_0$

$W_1(x) = a_1 x^3 + b_1 x + c_1 x + d_1$

Now,

$W_i''(x) = 6a_i x + 2b_i \quad$ for $i = 1, 2$

$W_i'''(x) = 6a_i$

$W_i''(0) = 0 = 2b_i \quad b_i = 0$

$W'''(0) = 0 = 6a_i \quad a_i = 0$

$\longrightarrow$ I can't use $W_i''(L) = W_i'''(L) = 0$ since they are automatically satisfied

$W_i(x) = c_i x + d_i$

Now for $w_0(x)$ that is a pure translation

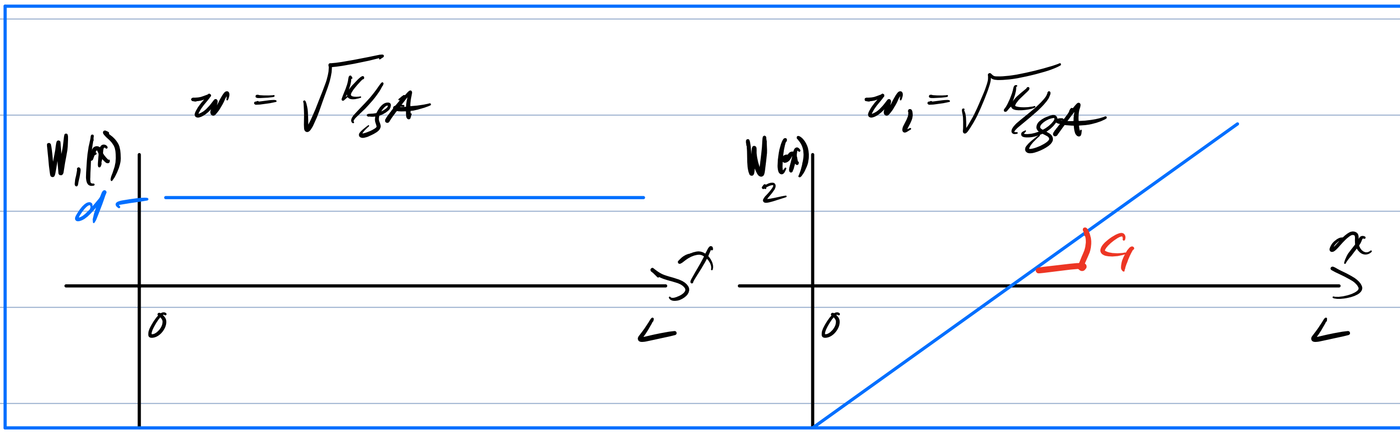
$w_1(x) = d$ just a constant that is scaled by initial condition

Now for $w_1(x)$ that is rotation about CG

$$W_2(x) = c_1 x + d_1$$

$$W_2(L/2) = 0 = c_1 L/2 + d \longrightarrow d = -c_1 L/2$$

$$W_2(x) = c_1(x - L/2)$$



Rigid Body Modes