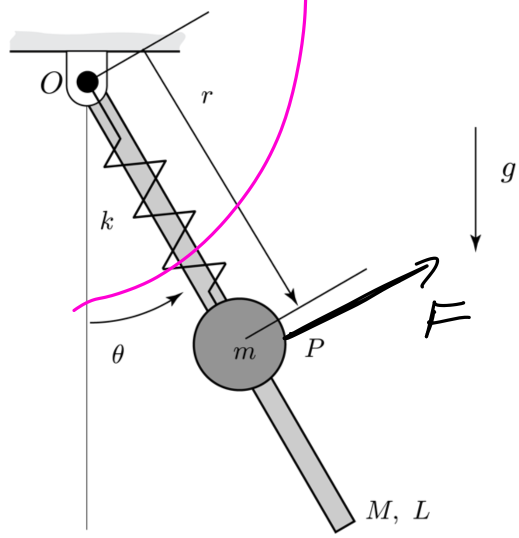
θ

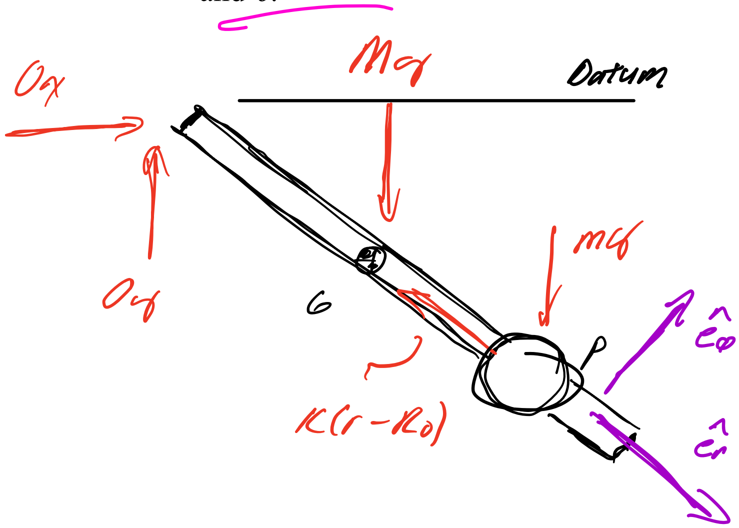
ME 563 - Fall 2022
Homework Problem 2.2

A thin, homogenous bar having a mass of M and length of L is pinned to the ground at point O. A particle P of mass m is free to slide on the smooth surface of the bar. A spring of stiffness k and unstretched length of R_0 is attached at point O and particle P. Let r be the radiald distance from O to P and Q be the rotation of the bar from a fixed vertical line.

a) Use Lagrange's equation to develop the EOM's for this two-DOF system using generalized coordinates of r and θ. Recall that the potential enrgy stored in a spring is related to the square of th stretch in the spring, where the stretch is equal to the difference between its actual length, and the unstretched length. Also, in writing down the velocity of vector of P, you may want to review the polar kinematic expressions for velocity.



b) Usin the equations of motion, determine the equilibrium values for r and θ.



$\vec{v}_1 = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

$\vec{v}_G = L/2\,\dot{\theta}\hat{e}_\theta$

$T = \frac{1}{2} m \vec{v}_1 \cdot \vec{v}_1 + \frac{1}{2} I_O \dot{\theta}^2$

$= \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2} I_O \dot{\theta}^2$

$I_O = \frac{1}{3} ML^2$

$U = -Mg\,L/2\cos\theta - mg(r)\cos\theta + \frac{1}{2}k(r-R_0)^2$

$T_{particle} = \cancel{\frac{1}{2} I_P \dot{\theta}^2}$

$I_P = mr^2$ r is fixed

$\dot{r}$ is present here

Laqrange's Equations

$\frac{d}{dt}\left(\partial T/\partial \dot{r}\right) - \partial T/\partial r + \partial U/\partial r = 0$

$$\partial T/\partial \dot r = m\dot r \rightarrow \frac{d}{dt}\left(\partial T/\partial \dot r\right) = m\ddot r$$

$$\partial T/\partial r = mr\dot\theta^2 \qquad \partial U/\partial r = -mg\cos\theta + k(r-R_0)$$

$$\underline{m\ddot r + mr\dot\theta^2 + k(r-R_0) - mg\cos\theta = 0}$$

$$\frac{d}{dt}\left(\partial T/\partial\dot\theta\right) - \partial T/\partial\theta + \partial U/\partial\theta = 0$$

$$\partial T/\partial\dot\theta = mr^2\dot\theta + I_0\dot\theta, \quad \frac{d}{dt}\left(\partial T/\partial\dot\theta\right) = mr^2\ddot\theta + I_0\ddot\theta + 2mr\dot r\dot\theta$$

$$\partial T/\partial\theta = 0, \quad \partial U/\partial\theta = \underline{mgr\sin\theta} + MgL/2\sin\theta$$

$$\underline{(mr^2 + 1/3ML^2)\ddot\theta + 2mr\dot r\dot\theta + g(mr + ML/2)\sin\theta = 0}$$

$$\boxed{\begin{aligned} &m\ddot r - mr\dot\theta^2 + k(r-R_0) - mg\cos\theta = 0 \\ &(mr^2 + 1/3ML^2)\ddot\theta + 2mr\dot r\dot\theta + g(mr + ML/2)\sin\theta = 0 \end{aligned}}$$

$$\ddot\theta = \dot\theta = \ddot r = \dot r = 0 \qquad r = r_{eq} \qquad \theta = \theta_{eq}$$

$$k(r_{eq} - R_0) - mg\cos\theta_{eq} = 0$$

$$g(mr_{eq} + ML/2)\sin\theta_{eq} = 0$$

$$\theta_{eq} = n\pi \qquad n = 0, 1, 2, 3, \ldots\ldots \infty$$

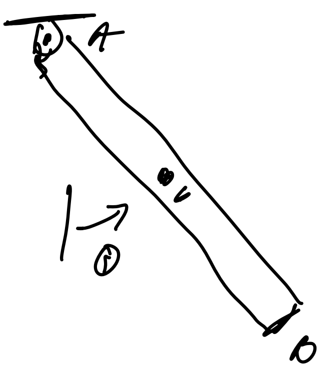
$$\cos\theta_{eq} = (-1)^n$$

$$\boxed{\begin{aligned} &k(r_{eq} - R_0) - mg(-1)^n = 0 \\ &r_{eq} = \frac{R_0 + (-1)^n mg}{k}, \quad \theta_{eq} = n\pi \\ &n = 0, 1, 2, 3, \ldots \infty \end{aligned}}$$

$$\left.\frac{\partial U}{\partial r}\right|_{q_{eq}} = mgr\sin\theta_{eq} + Mg\tfrac{L}{2}\sin\theta_{eq} = (mr + M\tfrac{L}{2})g\sin\theta_{eq} = 0$$

$$\left.\frac{\partial U}{\partial \theta}\right|_{q_e} = -mg\cos\theta_{eq} + k(r_{eq} - l_0) = 0$$

L' approximation is a Taylor Series around $\theta = \theta_{eq}$



Bar in fixed axis rotation about point A

$$T = \tfrac{1}{2} m \vec{V}_G \cdot \vec{V}_G + \tfrac{1}{2} I_G \dot{\theta}^2$$

$$= \tfrac{1}{2} I_A \dot{\theta}^2$$

regardless of coordinate system

$$\vec{V}_G = \frac{L}{2}\dot{\theta}\hat{e}_\theta$$

$$T = \tfrac{1}{2} m (\tfrac{L}{2}\dot{\theta}) \tfrac{L}{2}\dot{\theta} + \tfrac{1}{2}(\tfrac{1}{12} m L^2)\dot{\theta}^2$$

$$I_A = I_G + m d_{G/A}^2$$

$$= \tfrac{1}{12} mL^2 + m(\tfrac{L}{2})^2$$

$$= \tfrac{1}{3} mL^2$$

$$= \tfrac{1}{2} m \tfrac{L^2}{4}\dot{\theta}^2 + \tfrac{1}{2}\tfrac{1}{12} mL^2\dot{\theta}^2$$

$$= \tfrac{1}{2}(\tfrac{1}{3} mL^2)\dot{\theta}^2$$

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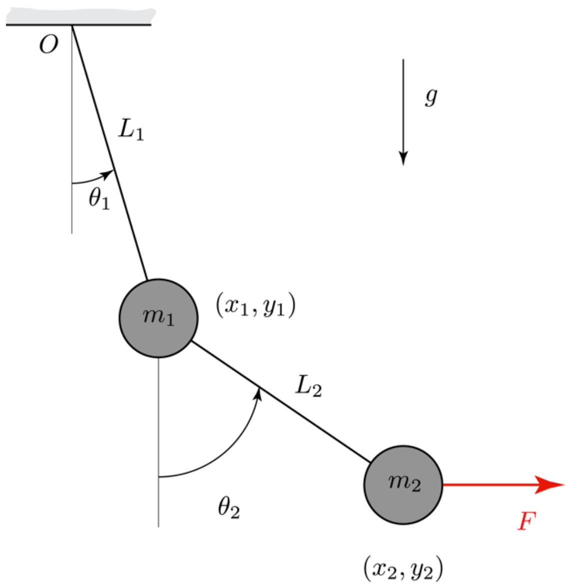
Homework Problem 2.3

A double pendulum consists of two bobs of mass m_1 and m_2, suspended by inextensible, massless strings of length L_1 and L_2.



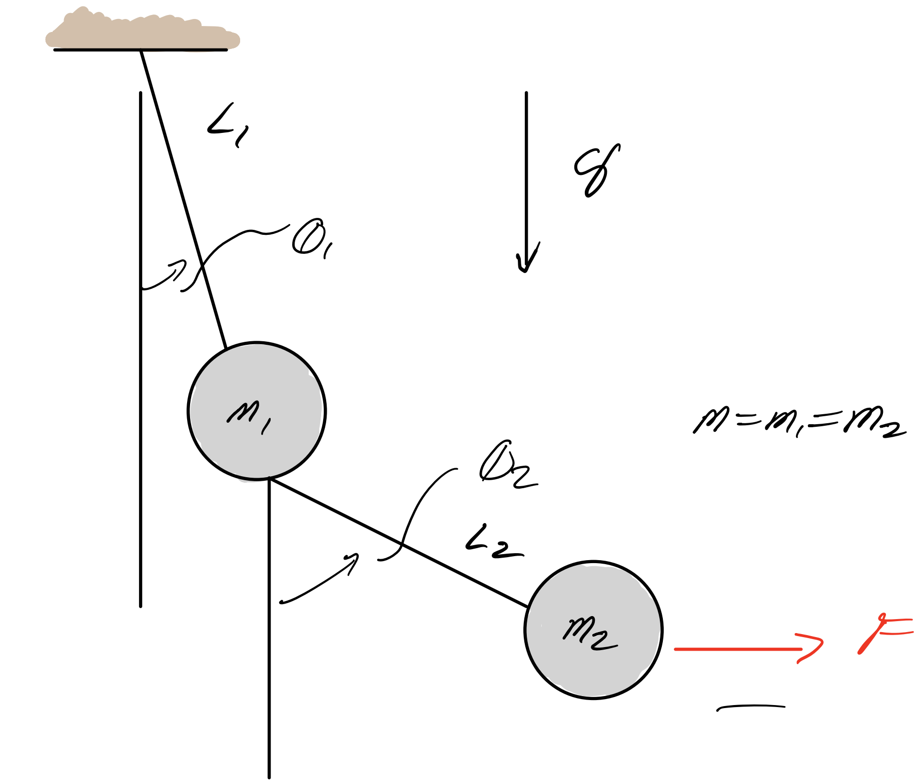
a) Determine the expression for potential energy U and the generalized forces corresponding to F for the generalized coordinates. Use these results to determine the angles θ_1 and θ_2 corresponding to static equilibrium. Leave these angles in terms of the ratio F/mg.

b) Write down an expression for the kinetic energy T in terms of the generalized coordinates θ_1 and θ_2 and their time derivatives. From this expression, identify the elements m_{ij} , where:

$$T = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} m_{ij}\dot{\theta}_i\dot{\theta}_j$$

c) Determine the mass matrix $[M]$ and the stiffness matrix $[K]$ corresponding small oscillations about the equilibrium state for $F/mg = 0$.

d) Determine the mass matrix $[M]$ and the stiffness matrix $[K]$ corresponding small oscillations about the equilibrium state for $F/mg = 2$. Compare these with those found in part c).



a) $U = -mgL_1\cos\theta_1 \quad -mg(L_1\cos\theta_1 \; + L_2\cos\theta_2)$

$= \; - \; 2mgL_1\cos\theta_1 \; - mgL_2\cos\theta_2$

$\vec{r}_2 = (L_1\sin\theta_1 \; + L_2\sin\theta_2)\hat{i} \; - (L_1\cos\theta_1 \; + L_2\cos\theta_2)\hat{j}$

$\delta\vec{r}_2 = (L_1\cos\theta_1\,\delta\theta_1 \; + L_2\cos\theta_2\,\delta\theta_2)\hat{i} \; + (L_1\sin\theta_1\,\delta\theta_1 + L_2\sin\theta_2\,\delta\theta_2)\hat{j}$

$\delta W^{(nc)} = \vec{F} \cdot \delta\vec{r}_2$

$= F\hat{i} \cdot \{ (L_1\cos\theta_1\,\delta\theta_1 \; + L_2\cos\theta_2\,\delta\theta_2)\hat{i} + (L_1\sin\theta_1\,\delta\theta_1 + L_2\sin\theta_2\,\delta\theta_2)\hat{j} \}$

$= FL_1\cos\theta_1\,\delta\theta_1 \quad + FL_2\cos\theta_2\,\delta\theta_2$

$Q_1 = FL_1\cos\theta_1$ and $Q_2 = FL_2\cos\theta_2$

Determine equilibrium points

$\frac{\partial U}{\partial \theta_1} = Q_1 \longrightarrow 2mgL_1\sin\theta_1 = FL_1\cos\theta_1$

$2mg\sin\theta_1 = F\cos\theta_1$

$2\tan\theta_1 = F/mg \longrightarrow \tan\theta_1 = F/2mg$

$$\theta_{1eq} = \tan^{-1}(F/2mg)$$

and

$$\frac{\partial U}{\partial \theta_2} = 0 \longrightarrow mgL_2\sin\theta_2 = FL_2\cos\theta_2$$

$$mg\sin\theta_2 = F\cos\theta_2$$
$$\tan\theta_2 = F/mg$$
$$\theta_{2eq} = \tan^{-1}(F/mg)$$

b) write Kinetic Energy

$$\vec{r}_1 = x_1\hat{i} + y_1\hat{j} = L_1\sin\theta_1\hat{i} - L_1\cos\theta_1\hat{j}$$

$$\vec{r}_2 = \vec{r}_1 + \vec{r}_{2/1} = (L_1\sin\theta_1 + L_2\sin\theta_2)\hat{i} - (L_1\cos\theta_1 + L_2\cos\theta_2)\hat{j}$$

The velocity vectors ...

$$\vec{v}_1 = \dot{\theta}_1 L_1\cos\theta_1\hat{i} + \dot{\theta}_1 L_1\sin\theta_1\hat{j} \quad \text{and} \quad \vec{v}_2 = (\dot{\theta}_1 L_1\cos\theta_1 + \dot{\theta}_2 L_2\cos\theta_2)\hat{i} + (\dot{\theta} L_1\sin\theta_1 + \dot{\theta}_2 L_2\sin\theta_2)\hat{j}$$

The kinetic energy ......

$$T = \tfrac{1}{2}m\vec{v}_1\cdot\vec{v}_1 + \tfrac{1}{2}m\vec{v}_2\cdot\vec{v}_2 = \tfrac{1}{2}(2mL_1^2\dot{\theta}_1^2 + 2mL_1L_2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2 + mL_2^2\dot{\theta}_2^2)$$

$$m_{11} = 2mL_1^2$$
$$m_{12} = m_{21} = mL_1L_2\cos(\theta_1-\theta_2)$$
$$m_{22} = mL_2^2$$

Stiffness Calculation

$$k_{11} = \left.\frac{\partial^2 U}{\partial\theta_1^2}\right|_{\bar{\theta}_{eq}} = 2mgL_1\cos(\theta_{1eq}), \quad k_{12} = k_{21} = \left.\frac{\partial^2 U}{\partial\theta_1\partial\theta_2}\right|_{\bar{\theta}_{eq}} = 0$$

$$k_{22} = \left.\frac{\partial^2 U_2}{\partial\theta_2^2}\right|_{\bar{\theta}_{eq}} = mgL_2\cos(\theta_{2eq})$$

c) Consider $F/mg = 0$ and $F/2mg = 0$

$$\theta_{1eq} = \theta_{2eq} = \tan^{-1}(0) = 0$$
$$\cos(\theta_{1eq}) = \cos(\theta_{2eq}) = 1$$

The linear mass and stiffness matrices

$$[M] = \begin{bmatrix} 2mL_1^2 & mL_1L_2 \\ mL_1L_2 & mL_2^2 \end{bmatrix}, \quad [K] = mg\begin{bmatrix} 2L_1 & 0 \\ 0 & L_1 \end{bmatrix}$$

d) Consider $F/mg = 2$ and $F/2mg = 1$

$\theta_{1eq} = \tan^{-1}(1) = \pi/4$

$\theta_{2eq} = \tan^{-1}(2) = 1.0715$

$\cos(\theta_{1eq}) = 1/\sqrt{2}$

$\cos(\theta_{2eq}) = 0.447$

The linear mass and stiffness matrices

$$[M] = \begin{bmatrix} 2\,mL_1^2 & 0.449\,mL_1L_2 \\ 0.449\,mL_1L_2 & mL_2^2 \end{bmatrix}, \quad [K] = mg\begin{bmatrix} 1.414\,L_1 & 0 \\ 0 & 0.447\,L_2 \end{bmatrix}$$

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Homework Problem 2.4

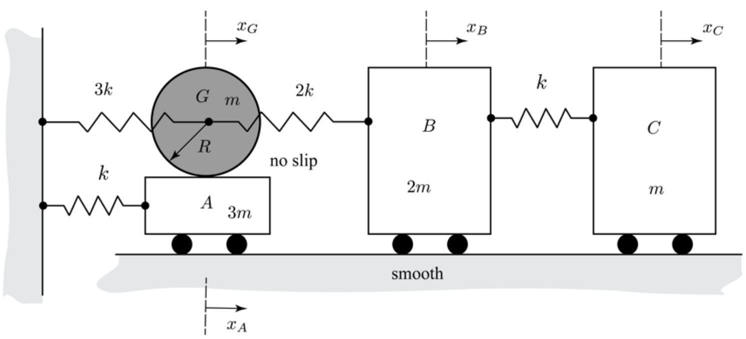
Consider the system below, whose motion is described by the absolute coordinates shown.

a) Write down the potential energy function U for this four-DOF system and use the following results from lecture to develop the stiffness matrix for the system:

$$K_{ij} = \left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\mathbf{q}_0}$$

b) Use the method of influence coefficients to develop the flexibility matrix $[A] = [K]^{-1}$.

c) Check your results in a) and b) above by verifying that $[A][K]=[I]$ where $[I]$ is the identity matrix.



a) The potential energy can be written as

$$U = \tfrac{1}{2} k x_A^2 + \tfrac{1}{2}(3k) x_G^2 + \tfrac{1}{2}(2k)(x_B - x_G)^2 + \tfrac{1}{2} k (x_C - x_B)^2$$

Equilibrium position : $x_A = x_G = x_B = x_C = 0$

$$\partial U/\partial x_A = k x_A, \quad \partial U/\partial x_G = 3k x_G + 2k(x_G - x_B) = 5k x_G - 2k x_B$$

$$\partial U/\partial x_B = 2k(x_B - x_G) + k(x_B - x_C) = 3k x_B - 2k x_G - k x_C$$

$$\partial U/\partial x_C = k x_C \qquad x_A = q_1, \; x_G = q_2, \; x_B = q_3, \; x_C = q_4$$

$$K_{11} = \frac{\partial^2 U}{\partial x_A^2} = k, \quad K_{12} = K_{21} = \frac{\partial^2 U}{\partial x_A \partial x_G} = 0, \quad K_{13} = K_{31} = \frac{\partial^2 U}{\partial x_A \partial x_B} = 0, \quad K_{14} = \frac{\partial^2 U}{\partial x_A \partial x_C} = 0$$

$k_{22} = \frac{\partial^2 U}{\partial x_b^2} = 5k$, $\quad k_{23} = k_{32} = \frac{\partial^2 U}{\partial x_b \partial_b} = -2k$, $\quad k_{24} = k_{42} = \frac{\partial^2 U}{\partial x_b \partial x_c} = 0$

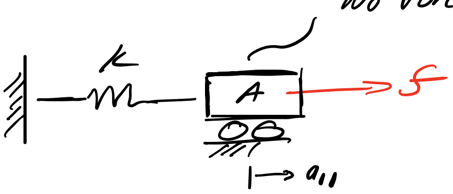
$k_{33} = \frac{\partial^2 U}{\partial^2 x_B} = 3k \qquad k_{34} = k_{43} = \frac{\partial^2 U}{\partial x_B \partial x_c} = -k$

$k_{44} = \frac{\partial^2 U}{\partial^2 x_c} = k$

$$[K] = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 5k & -2k & 0 \\ 0 & -2k & 3k & -k \\ 0 & 0 & -k & k \end{bmatrix}$$

b) Apply a unit load @ A

No force here

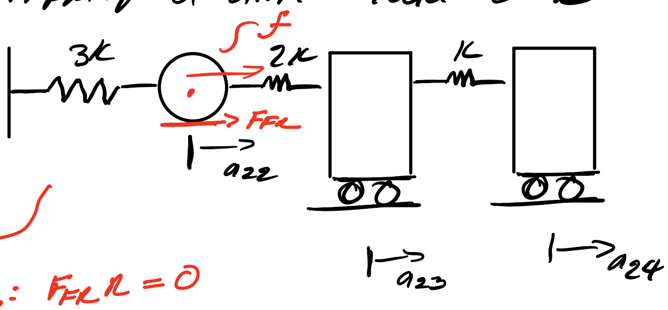


$\Sigma F_x: \; -k a_{11} + \cancel{f}^{\,1} = 0, \quad a_{11} = 1/k$

$a_{21} = a_{31} = a_{41} = 0$ since not connected

By reciprocity $a_{12} = a_{21} = 0$, $a_{13} = a_{31} = 0$, $a_{41} = a_{14} = 0$

Apply a unit load @ B



$\curvearrowleft \Sigma M_o: \; F_{FR} R = 0$

$F_{FR} = 0$

$a_{22} = a_{23} = a_{24}$ zero strain to right

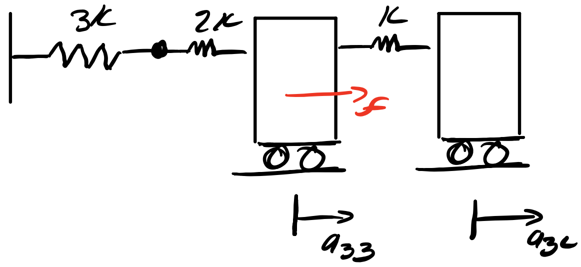
$\Sigma F_x: \; -3k a_{22} + \cancel{f}^{\,1} + \cancel{F_{FR}}^{\,0} = 0$

$a_{22} = 1/3k, \qquad a_{24} = 1/3k$

$a_{23} = 1/3k,$

By reciprocity $a_{42} = a_{24} = 1/3k$, $a_{23} = a_{32} = 1/3k$

Apply a unit load @ C



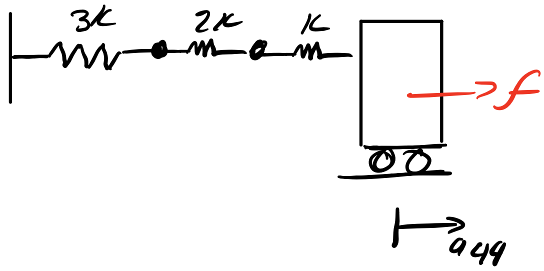
$1/k_{eq} = 1/3k + 1/2k$

$k_{eq} = \frac{(3k)(2k)}{3k+2k} = \frac{6}{5}k$

$\Sigma F_x : -k_{eq} a_{33} + f^{1} = 0$ $\qquad a_{33} = 1/k_{eq} = 5/6k$

$a_{34} = a_{43} = 5/6k$

Apply a unit load @ D



$1/k_{eq1} = 1/3k + 1/2k, \quad k_{eq1} = \frac{(3k)(2k)}{3k+2k} = \frac{6}{5}k$

$1/k_{eq2} = 1/k_{eq1} + 1/k = 5/6k + 1/k$

$k_{eq2} = \frac{k_{eq1} k}{k_{eq1} + k} = \frac{(6/5 k)(k)}{6/5 k + k}$

$k_{eq2} = \frac{6/5 k^2}{11/5 k} = 6/11\, k$

$\Sigma F_x : -k_{eq2} a_{44} + 1 = 0$

$a_{44} = 1/k_{eq2} = 11/6k$

$$[a] = \begin{bmatrix} 1/k & 0 & 0 & 0 \\ 0 & 1/3k & 1/3k & 1/3k \\ 0 & 1/3k & 5/6k & 5/6k \\ 0 & 1/3k & 5/6k & 11/6k \end{bmatrix}$$

$$[a][k] = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & 5k & -2k & 0 \\ 0 & -2k & 3k & -k \\ 0 & 0 & -k & k \end{bmatrix} \begin{bmatrix} 1/k & 0 & 0 & 0 \\ 0 & 1/3k & 1/3k & 1/3k \\ 0 & 1/3k & 5/6k & 5/6k \\ 0 & 1/3k & 5/6k & 11/6k \end{bmatrix}$$

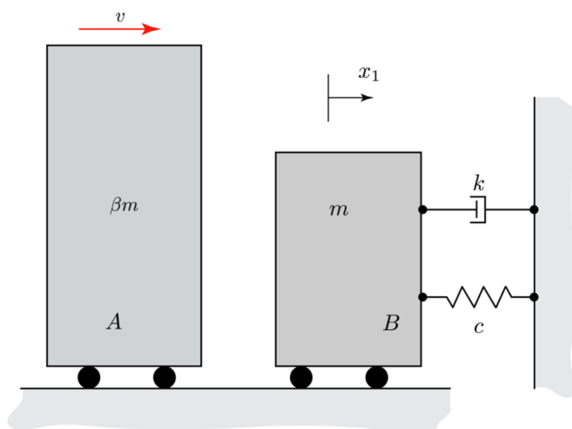
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

since $[k]^{-1}[k] = [I]$

therefore $[a] = [k]^{-1}$

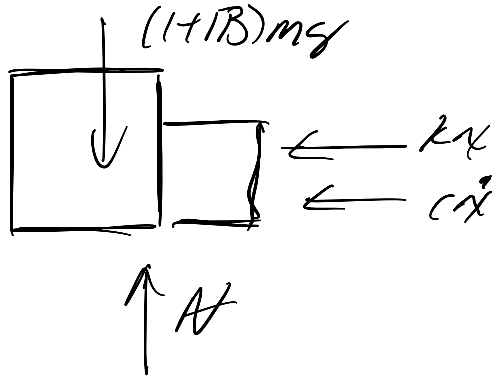
ME 563 - Fall 2020
Homework Problem 4.1

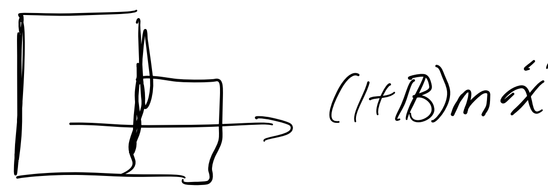
Block A is initially moving to the right on a smooth, horizontal surface with a speed of v, with B at rest and the spring unstretched. Upon impact, at time $t = 0$, block A sticks to block B. For this problem use: $v = 10\text{m/ sec}$, $m = 10$ kg , $c = 60$ kg / sec and $k = 300$ N / m.



a) Derive the differential equation of motion (EOM) for the system corresponding to $t > 0$.
b) Determine the numerical values for the undamped natural frequency and the damping ratio of the system corresponding to the EOM found in a) when $\beta=1$. How does the value change when $\beta>1$ and $\beta<1$.
c) Determine the response of the system for $t > 0$. HINT: You can use conservation of momentum to determine the speed of blocks A and B immediately after sticking when $\beta=1$. How does the value change when $\beta>1$ and $\beta<1$.
d) What is the maximum displacement of blocks A and B in the response found in c)? when $\beta=1$. How does the value change when $\beta>1$ and $\beta<1$.

a)





$$\overset{+}{\rightarrow}\Sigma F_x : -kx - c\dot{x} = m(1+B)\ddot{x}$$

$$m(1+B)\ddot{x} + c\dot{x} + kx = 0$$

$$\ddot{x} + \frac{c\dot{x}}{m(1+B)} + \frac{k}{m(1+B)}x = 0$$

b) $$\omega_n = \sqrt{\frac{k}{m(1+B)}}$$

$$2\zeta\omega_n = \frac{c}{m(1+B)} \longrightarrow \zeta = \frac{c}{2m(1+B)\omega_n}$$

$$\zeta = \frac{c}{2m(1+B)\sqrt{\frac{k}{m(1+B)}}} = \frac{c}{2\sqrt{km(1+B}}$$

$$\omega_D = \omega_n\sqrt{1-\zeta^2} = \sqrt{\omega_n^2(1-\zeta^2)}$$

$$\omega_D = \sqrt{\frac{k}{m(1+B)}\left(1 - \frac{c^2}{4km(1+B)}\right)}$$

@ $B=1$

$\omega_n = 3.873$ rad/s

$\zeta = 0.3873$

$\omega_D = 3.5107$ rad/s

c) Initially velocity

$$BmV = m(1+B)\dot{x}(0) \longrightarrow \dot{x}(0) = \frac{B}{(1+B)}V$$

write governing equation

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n x = 0$$

Solution

$$x(t) = e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$

the velocity

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(A\cos\omega_d t + B\sin\omega_d t)$$
$$+ e^{-\zeta\omega_n t}(-\omega_D A\sin\omega_D t + \omega_D B\cos\omega_D t$$

evaluate IC

$$x(0) = A = 0$$
$$\dot{x}(0) = -\zeta\omega_n(A) + (1)\omega_D B$$

$$B = \frac{\dot{x}(0)}{\omega_D}$$

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0)}{\omega_D}\sin\omega_D t\right)$$

d) Max displacement when $\dot{x}(t) = 0$

$$\dot{x}(t) = e^{-\zeta\omega_n t}\frac{\dot{x}(0)}{\omega_D}\left(-\zeta\omega_n\sin\omega_D t + \omega_D\cos\omega_D t\right)$$

$$\dot{x}(t_{max}) = 0 = e^{-\zeta\omega_n t_{max}}\frac{\dot{x}(0)}{\omega_D}\left(-\zeta\omega_n\sin\omega_D t + \omega_D\cos\omega_D t\right)$$

$$0 = -\zeta\omega_n\sin\omega_D t + \omega_D\cos\omega_D t$$

$$\zeta\omega_n\sin\omega_D t = \omega_D\cos\omega_D t$$

$$\frac{\sin \omega_D t}{\cos \omega_D t} = \frac{\omega_D}{\zeta \omega_n} = \frac{\omega_n \sqrt{1-\zeta^2}}{\zeta \omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

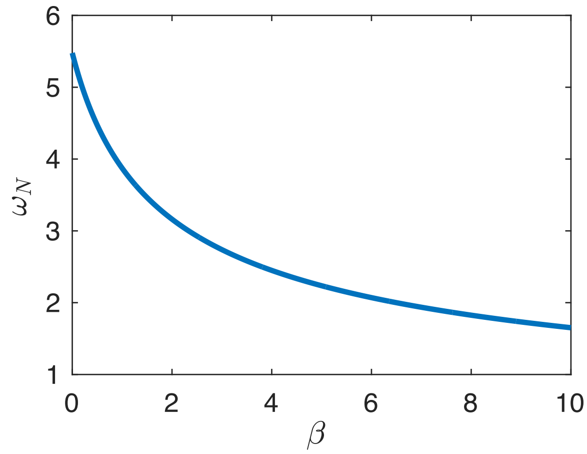
$$\tan \omega_D t = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\omega_D t_{max} = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

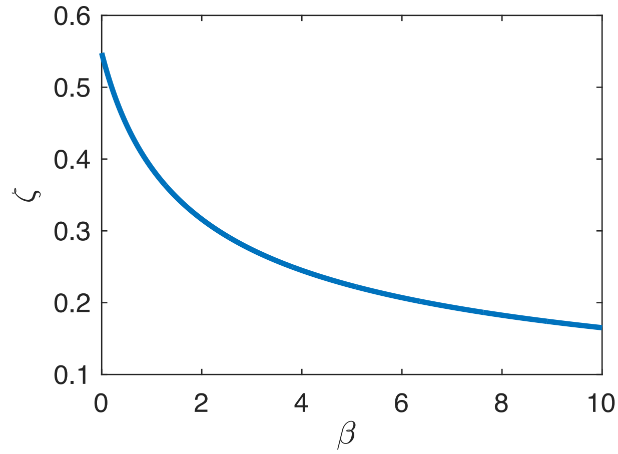
$$t_{max} = \frac{1}{\omega_D} \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Evaluate $x(t_{max})$

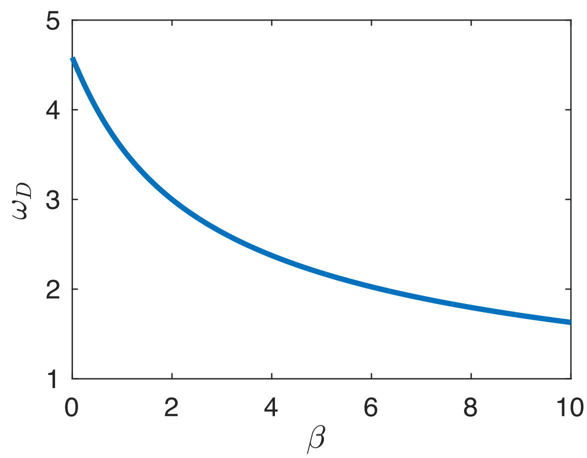
See plots



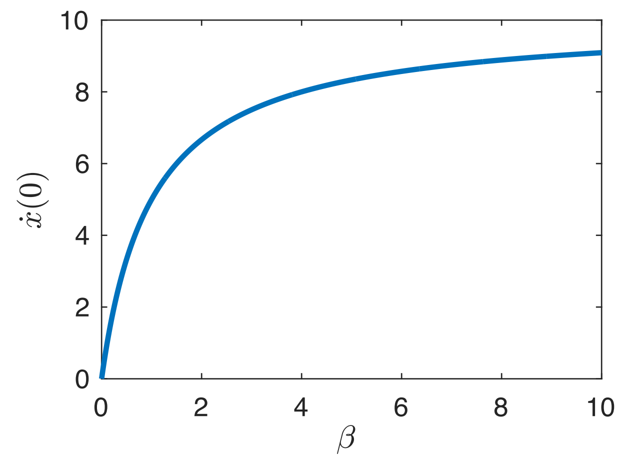
a)



b)

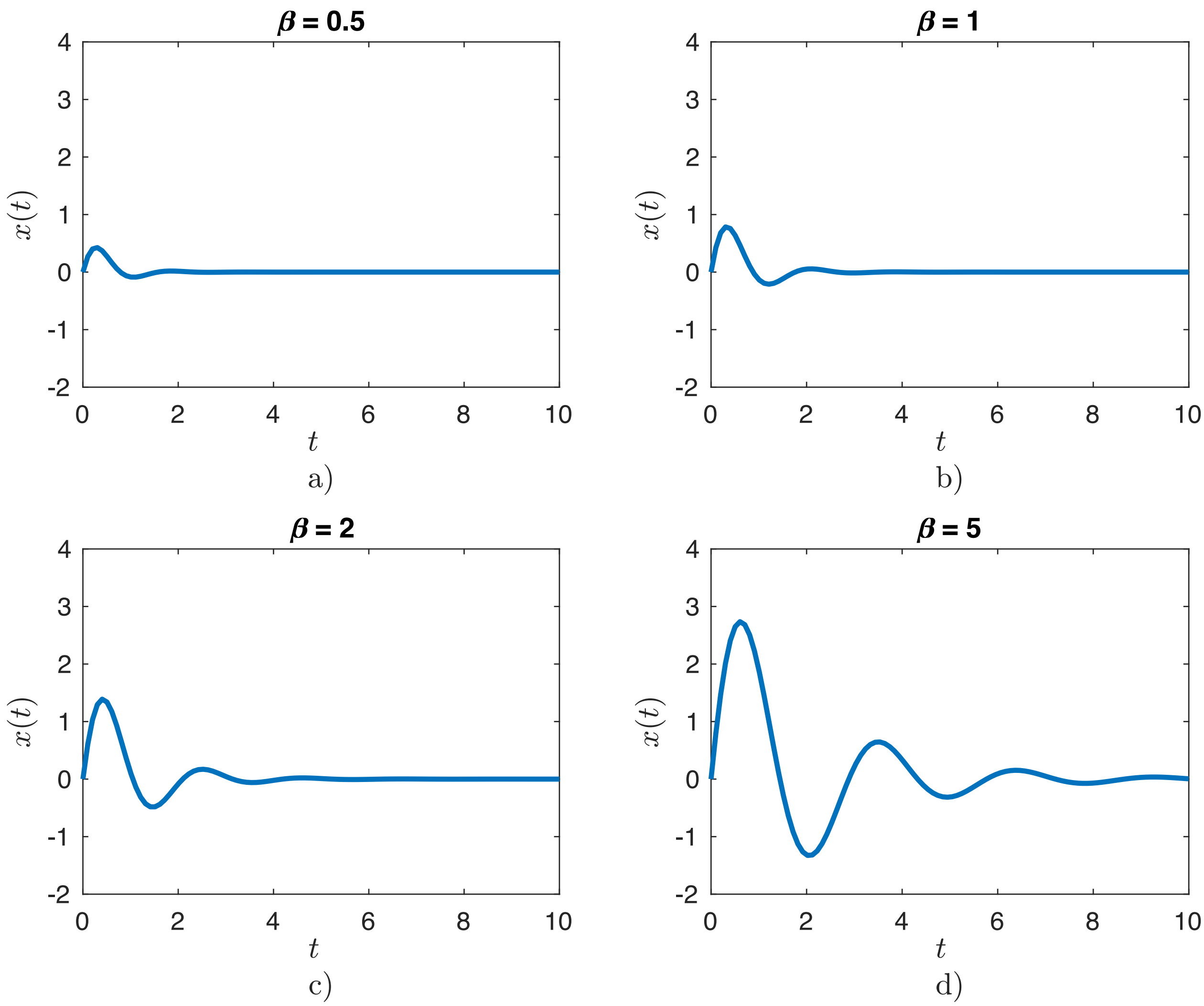


c)



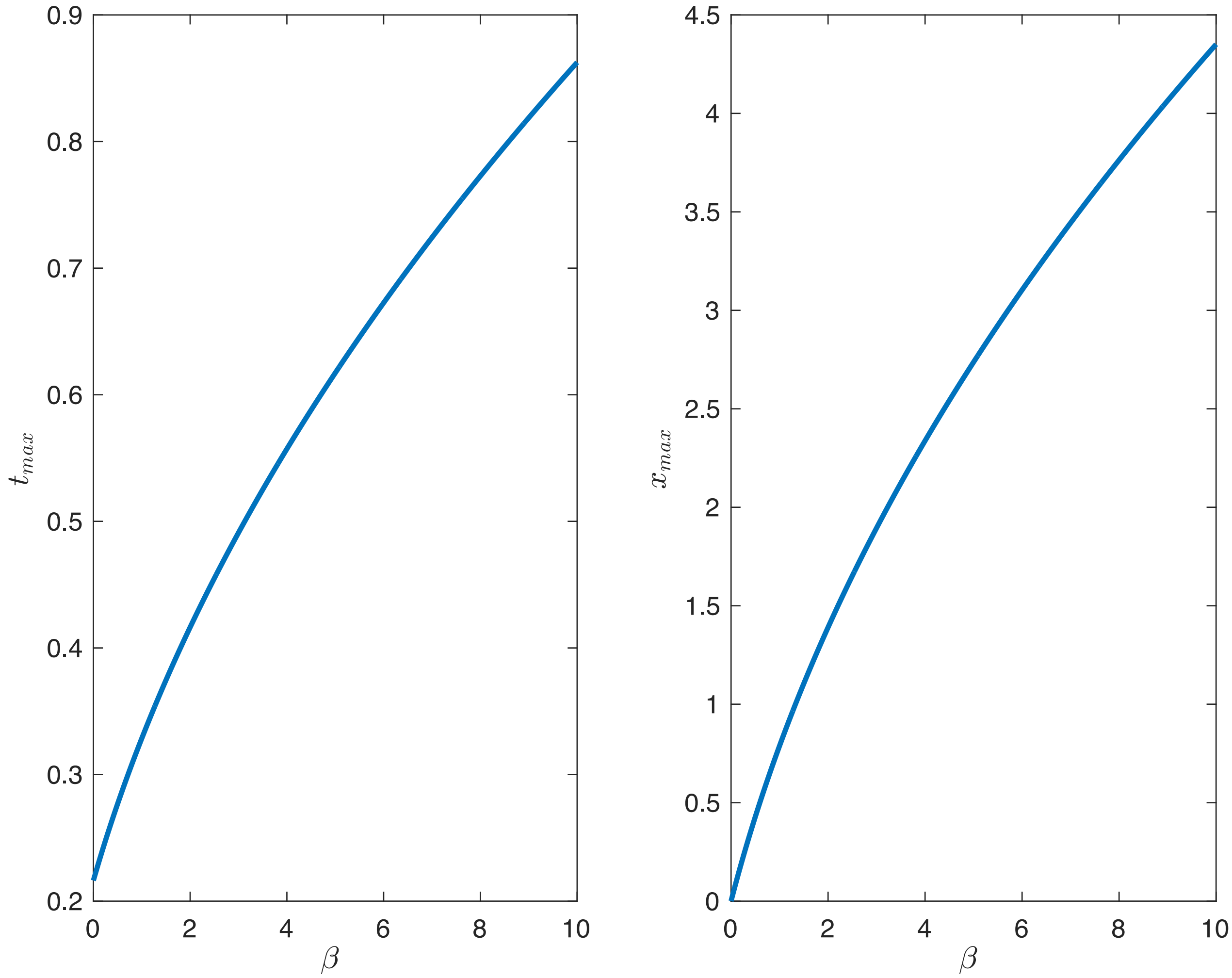
d)

Note that $\omega_N, \zeta, \omega_D$ decrease as β increases the initial velocity $\dot{x}(0)$ approaches v as β increases, an $\omega_N > \omega_D$ for all β.



max amplitude increases a β increases

as β increases, damping decrease as β increases



a) b)

Both t_{max}, x_{max} ↑ as β ↑, the more massive the object the longer it takes to deform and the deformation increases.

ME 563 - Fall 2020
Homework Problem 4.2

a) Show that the logarithmic decrement is equal to

$$\delta = \frac{1}{n}\ln\frac{x_0}{x_n}$$

where x_n is the amplitude of vibration after n cycles have elapsed.

b) Show that by calculation that

$$A\sin(\omega_n t + \phi)$$

can be represented as

$$B\sin(\omega_n t) + C\cos(\omega_n t)$$

where B and C are functions of A and ϕ.

c) Solve

$$\ddot{x} - \dot{x} + x = 0$$

with initial conditions $x(0)=1$, and $v(0)=0$ for x(t) and sketch the time waveform.

1) Show that $\delta = \frac{1}{n} \ln \frac{x_0}{x_n}$

Recall, $x_0/x_1 = e^{2\pi\zeta/\sqrt{1-\zeta^2}}$

then $x_0/x_1 \cdot x_1/x_2 \cdot \cdot \cdot x_N/x_{N+1} = \left(e^{2\pi\zeta/\sqrt{1-\zeta^2}}\right)^N$

$$x_0/x_N = e^{N 2\pi\zeta/\sqrt{1-\zeta^2}}$$

$$\ln\left(x_0/x_N\right) = \cancel{\frac{N 2\pi\zeta}{\sqrt{1-\zeta^2}}}\ \delta = N\delta$$

Rearranging, $\delta = \frac{1}{n} \ln\left(\frac{x_0}{x_N}\right)$

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2) $A\sin(\omega_n t + \phi)$

Now, $A\sin(\omega_n t + \phi) = A\sin\omega_n t\cos\phi + A\cos\omega_n t\sin\phi$

$$= A\cos\phi\sin\omega_n t + A\sin\phi\cos\omega_n t$$

where $B = A\cos\phi$, $C = A\sin\phi$

then $A\sin(\omega_n t + \phi) = B\sin\omega_n t + C\cos\omega_n t$

Note that $A^2 = B^2 + C^2$ or $A = \sqrt{B^2 + C^2}$

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3) Solve $\ddot{x} - \dot{x} + x = 0$ w $x(0) = 1$ and $\dot{x}(0) = 0$

assume $x(t) = Ae^{\lambda t}$, $\dot{x}(t) = \lambda Ae^{\lambda t}$, $\ddot{x}(t) = \lambda^2 Ae^{\lambda t}$

$$\ddot{x} - \dot{x} + x = (\lambda^2 - \lambda + 1)Ae^{\lambda t} = 0$$

CE: $\lambda^2 - \lambda + 1 = 0 \longrightarrow \lambda_{1,2} = 0.5 \pm 0.86 j$

$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$ and $\dot{x}(t) = \lambda_1 Ae^{\lambda_1 t} + \lambda_2 Be^{\lambda_2 t}$

using initial conditions . . . .

$x(0) = x_0 = A + B$ and $\dot{x}(0) = 0 = \lambda_1 A + \lambda_2 B$

$$B = -\frac{\lambda_1}{\lambda_2} A \longrightarrow x_0 = A - \lambda_1/\lambda_2 \, A$$

$$A = \left(\frac{1}{1 - \lambda_1/\lambda_2}\right) x_0 \quad \text{and} \quad B = \frac{-\lambda_1/\lambda_2}{(1 - \lambda_1/\lambda_2)} x_0$$

$$x(t) = \left(\frac{1}{1 - \lambda_1/\lambda_2}\right) x_0 e^{\lambda_1 t} + \left(\frac{-\lambda_1/\lambda_2}{1 - \lambda_1/\lambda_2}\right) x_0 e^{\lambda_2 t}$$