Problem 4.1

Position vectors

$\vec{r}_1 = x_1 \hat{\imath}$, $\vec{r}_2 = x_2 \hat{\imath}$ and $\vec{r}_3 = x_3 \hat{\imath}$

$\vec{v}_1 = \dot{x}_1 \hat{\imath}$, $\vec{v}_2 = \dot{x}_2 \hat{\imath}$ and $\vec{v}_3 = \dot{x}_3 \hat{\imath}$

The kinetic energy

$$T = \tfrac{1}{2} m \vec{v}_1 \cdot \vec{v}_1 + \tfrac{1}{2} \beta m \vec{v}_2 \cdot \vec{v}_2 + \tfrac{1}{2} m \vec{v}_3 \cdot \vec{v}_3 = \tfrac{1}{2} \underbrace{m}_{m_{11}} \dot{x}_1^2 + \tfrac{1}{2} \underbrace{\beta m}_{m_{22}} \dot{x}_2^2 + \tfrac{1}{2} \underbrace{m}_{m_{33}} \dot{x}_3^2$$

$$[m] = \begin{bmatrix} m & 0 & 0 \\ 0 & \beta m & 0 \\ 0 & 0 & m \end{bmatrix}$$

The potential energy

$$U = \tfrac{1}{2} k (x_2 - x_1)^2 + \tfrac{1}{2} k (x_3 - x_2)^2$$

$$\frac{\partial U}{\partial x_1} = k(x_1 - x_2)$$

$$\frac{\partial U}{\partial x_2} = k(x_2 - x_1) + k(x_2 - x_3) = k(2x_2 - x_1 - x_3)$$

$$\frac{\partial U}{\partial x_3} = k(x_3 - x_2)$$

Equilibrium position

$$x_{1e} = x_{2e} = x_{3e} = 0$$

$$k_{11} = \frac{\partial^2 U}{\partial x_1} = k , \quad k_{12} = k_{21} = \frac{\partial^2 U}{\partial x_1 \partial x_2} = -k , \quad k_{13} = k_{31} = \frac{\partial^2 U}{\partial x_1 \partial x_3} = 0$$

$$k_{22} = \frac{\partial^2 U}{\partial x_2^2} = 2k , \quad k_{23} = k_{32} = \frac{\partial^2 U}{\partial x_2 \partial x_3} = -k$$

$$k_{33} = \frac{\partial^2 U}{\partial x_3^2} = k$$

$$[k] = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$$

EOM: $[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0}$ assume $\vec{x} = \vec{X}e^{\lambda t}$

$$\left[\lambda^2[m] + [k]\right]\vec{X}\cancel{e^{\lambda t}} = 0$$

$$\begin{bmatrix} \lambda^2 m + k & -k & 0 \\ -k & \lambda^2\beta m + 2k & -k \\ 0 & -k & \lambda^2 m + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \vec{0}$$

Take determinant to determine CE

$$CE = \lambda^2 m + k \begin{vmatrix} \lambda^2 m^2 + 2k & -k \\ -k & \lambda^2\beta m + k \end{vmatrix}$$

$$- (-k) \begin{vmatrix} -k & -k \\ 0 & \lambda^2 m + k \end{vmatrix}$$

$$CE = (\lambda^2 m + k)\left((\lambda^2 m + 2k)(\lambda^2\beta m + k) - k^2\right)$$
$$+ k(-k)(\lambda^2 m + k)$$

$$CE = m^3\beta\lambda^6 + (\beta+1)2km^2\lambda^4 + (\beta+2)k^2 m\lambda^2 = 0$$

$$\lambda = i\omega$$

$$CE = \beta m^3\omega^6 - (\beta+1)2km^2\omega^4 + (\beta+2)k^2 m\omega^2 = 0$$

Rewrite

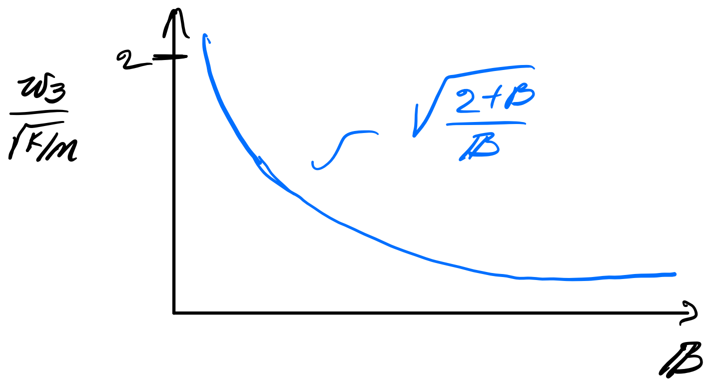
$$CE = \left(\beta m^3\omega^4 - (\beta+1)2km^2\omega^2 + (\beta+2)k^2 m\right)\omega^2 = 0$$

Factoring and only keeping positive roots

$\omega_1 = 0$, $\omega_2 = \sqrt{k/m}$, $\omega_3 = \left(\sqrt{\frac{2+\beta}{\beta}}\right)\sqrt{k/m}$

$\omega_1 = 0$ indicates a rigid Body mode

ω_3's behavior is interesting



as β increases the natural frequency decreases

Solve for modal vectors

$$\begin{bmatrix} -\omega^2 m + k & -k & 0 \\ -k & -\omega^2 \beta m + 2k & -k \\ 0 & -k & -\omega^2 m + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \vec{0}$$

Let $\omega = 0$

$$\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ X_2 \\ X_3 \end{Bmatrix} = \vec{0}$$

$$k - k X_2 = 0 \qquad X_2 = 1$$

$$-k(1) + 2k(1) - k X_3 = 0 \longrightarrow X_3 = 1$$

$$\bar{X}^1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$$\omega = \sqrt{k/m}$$

$$\begin{bmatrix} 0 & -k & 0 \\ -k & -\beta k + 2k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} 1 \\ X_2 \\ X_3 \end{Bmatrix} = \bar{0}$$

$$X_2 = 0$$

$$-k(1) + (-\beta k + 2k)0 - k X_3 = 0 \qquad X_3 = -1$$

$$\bar{X}^2 = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$$

$$\omega = \sqrt{\frac{(2+\beta)}{\beta}} \sqrt{k/m}$$

$$\begin{bmatrix} \overset{-2k/\beta}{-\frac{(2+\beta)}{\beta} k + k} & -k & 0 \\ -k & \left(\frac{-\beta(2+\beta)}{\beta}\right) k + 2k & -k \\ 0 & -k & \underset{-2k/\beta}{-\frac{(2+\beta)}{\beta} k + k} \end{bmatrix} \begin{Bmatrix} 1 \\ X_2 \\ X_3 \end{Bmatrix} = \bar{0}$$

$$-2k/B(1) \quad - k(X_2) = 0$$

$$X_2 = -2/B$$

$$-k(1) \quad \left(-(2+B)k + 2k\right)\left(-2/B\right) \quad - k X_3 = 0$$

$$-k + (-2k + 2k - Bk)(-2/B) \quad - k X_3 = 0$$

$$-k \quad + 2k \quad - k X_3 \quad \Rightarrow \quad X_3 = 1$$

In summary

$$X^{(i)} = \begin{Bmatrix} 1 \\ -\omega_i^2 m/k + 1 \\ 1 \end{Bmatrix}$$

$$\omega_1 = 0 \qquad \omega_2 = \sqrt{k/m} \qquad \omega_3 = \left(\sqrt{(2+B)/B}\right)\sqrt{k/m}$$

$$X^1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \qquad X^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \qquad X^3 = \begin{Bmatrix} 1 \\ -2/B \\ 1 \end{Bmatrix}$$

These mode shapes are orthogonal

$$x(t) = c_1 \bar{\bar{X}}^1 \cos w_1 t + s_1 \bar{\bar{X}}^1 \sin w_1 t +$$
$$c_2 \bar{\bar{X}}^2 \cos w_2 t + s_2 \bar{\bar{X}}^2 \sin w_2 t +$$
$$c_3 \bar{\bar{X}}^3 \cos w_3 t + s_3 \bar{\bar{X}}^3 \sin w_3 t$$

$$s_1 = s_2 = s_3 \rightarrow \text{if } \dot{\bar{x}}(0) = \bar{\bar{0}}$$

$$c_1 = \frac{\bar{\bar{X}}^{1T} [m] \bar{x}(0)}{\bar{\bar{X}}^{1T} [m] \bar{\bar{X}}^1} \qquad c_2 = \frac{\bar{\bar{X}}^{2T} [m] \bar{x}(0)}{\bar{\bar{X}}^{2T} [m] \bar{\bar{X}}^2}$$

$$c_3 = \frac{\bar{\bar{X}}^{3T} [m] \bar{x}(0)}{\bar{\bar{X}}^{3T} [m] \bar{\bar{X}}^3}$$

for a system that responds at the rigid body mode set $\bar{x}(0) = x_0 \bar{\bar{X}}^1 = \begin{Bmatrix} x_0 \\ x_0 \\ x_0 \end{Bmatrix}$

$$c_1 = \frac{\{1 \;\; 1 \;\; 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & Bm & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} x_0 \\ x_0 \\ x_0 \end{Bmatrix}}{\{1 \;\; 1 \;\; 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & Bm & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}} = \frac{x_0 m (2+B)}{m(2+B)}$$

$$c_1 = x_0$$

$$C_2 = \frac{\{1\ \ 0\ \ -1\} \begin{bmatrix} m & 0 & 0 \\ 0 & \mathbb{B}m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} x_0 \\ x_0 \\ x_0 \end{Bmatrix}}{\{1\ \ 0\ \ -1\} \begin{bmatrix} m & 0 & 0 \\ 0 & \mathbb{B}m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}} = \frac{0}{2m}$$

$$C_2 = 0$$

$$C_3 = \frac{\{1\ \ -2/\mathbb{B}\ \ 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & \mathbb{B}m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} x_0 \\ x_0 \\ x_0 \end{Bmatrix}}{\{1\ \ -2/\mathbb{B}\ \ 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & \mathbb{B}m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -2/\mathbb{B} \\ 1 \end{Bmatrix}} = \frac{0}{2m + 4m/\mathbb{B}}$$

then

$\underline{\bar{X}} = c_1 \vec{\bar{X}}^1 \cos w_1 t \longrightarrow$ only 1st mode responds.

for a system that responds with 1st flexural mode

set $\ddot{\bar{x}}_0 = x_0 \bar{X}^2$, $\dot{\bar{x}}_0 = \bar{0}$ $\rightarrow s_1 = s_2 = s_3 = 0$

$$= \begin{Bmatrix} x_0 \\ 0 \\ -x_0 \end{Bmatrix}$$

$$c_1 = \frac{\{1 \quad 1 \quad 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & Bm & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} x_0 \\ 0 \\ -x_0 \end{Bmatrix}}{\{1 \quad 1 \quad 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & Bm & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}} = \frac{0}{m(2+B)}$$

$$c_2 = \frac{\{1 \quad 0 \quad -1\} \begin{bmatrix} m & 0 & 0 \\ 0 & Bm & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} x_0 \\ 0 \\ -x_0 \end{Bmatrix}}{\{1 \quad 0 \quad -1\} \begin{bmatrix} m & 0 & 0 \\ 0 & Bm & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}} = \frac{2m x_0}{2m}$$

$c_2 = x_0$

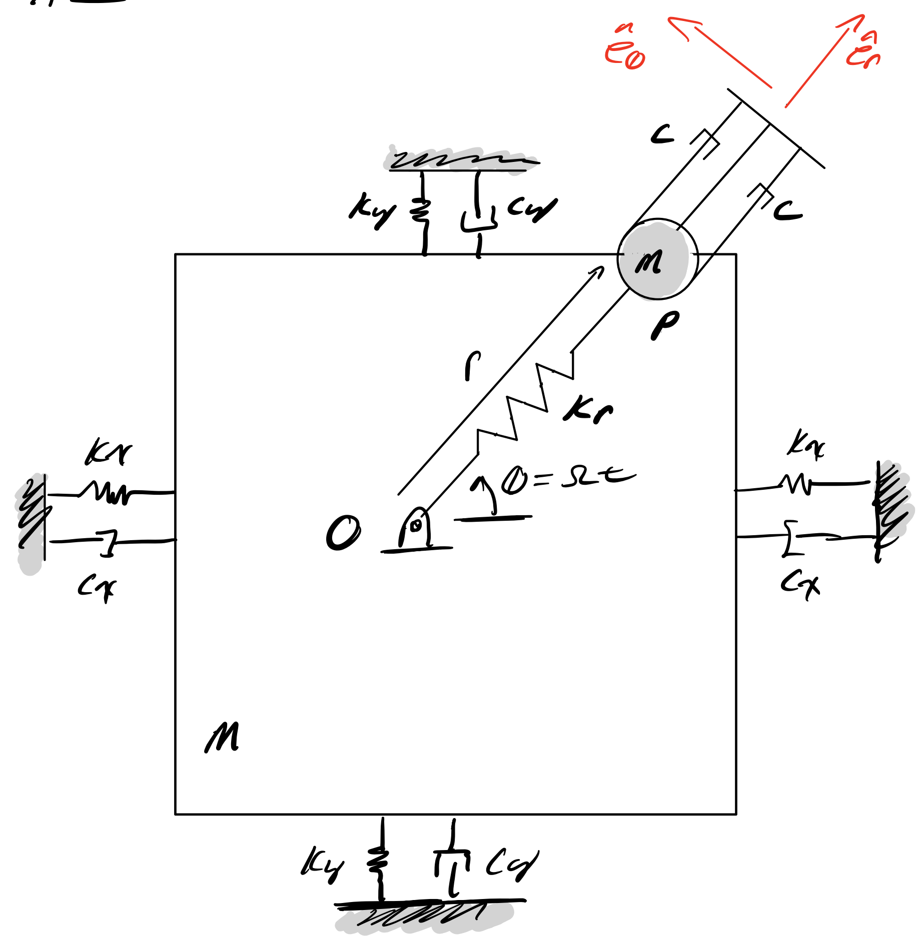
$$C_3 = \frac{\{1 \;\; -2/\sqrt{B} \;\; 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & \sqrt{B}m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} x_0 \\ 0 \\ -x_0 \end{Bmatrix}}{\{1 \;\; -2/\sqrt{B} \;\; 1\} \begin{bmatrix} m & 0 & 0 \\ 0 & \sqrt{B}m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} 1 \\ -2/\sqrt{B} \\ 1 \end{Bmatrix}} = \frac{0}{2m + 4m/\sqrt{B}}$$

$$C_3 = 0$$

$$\bar{x} = C_2 \bar{X}^2 \cos \omega_2 t$$

a 1 mode response

Problem 4.2



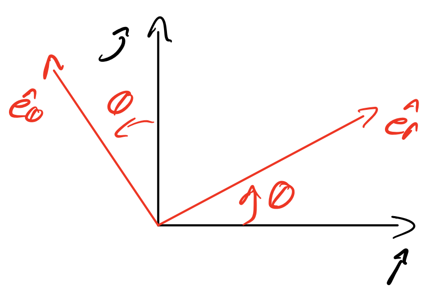
(1)

$$\vec{r}_o = x\hat{i} + y\hat{j}$$

$$\vec{r}_p = \vec{r}_o + \vec{r}_{p/o} = x\hat{i} + y\hat{j} + r\hat{e}_r$$

$$\vec{v}_p = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$\hat{i}, \hat{j}$ and $\hat{e}_r$ and $\hat{e}_\theta$ are two different coordinate systems but they are related by



$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$$

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

$$\vec{v}_p = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{r}\cos\theta\hat{i} + \dot{r}\sin\theta\hat{j} - r\dot{\theta}\sin\theta\hat{i} + r\dot{\theta}\cos\theta\hat{j}$$

$$\vec{v}_p = (\dot{x} + \dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{i} + (\dot{y} + \dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{j}$$

Write out kinetic & potential energy, rayliegh dissipation function

$$T = \tfrac{1}{2} M \dot{x}^2 + \tfrac{1}{2} M \dot{y}^2 + \tfrac{1}{2} m \vec{v}_p \cdot \vec{v}_p$$

$$T = \tfrac{1}{2} M \dot{x}^2 + \tfrac{1}{2} M \dot{y}^2 + \tfrac{1}{2} m (\dot{r} \sin\theta + \dot{y} + r\dot{\theta}\cos\theta)^2$$
$$+ \tfrac{1}{2} m (\dot{r}\cos\theta + \dot{x} - r\dot{\theta}\sin\theta)^2$$

$$T = \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2}(m+M)\dot{x}^2 + \tfrac{1}{2}(m+M)\dot{y}^2 + \tfrac{1}{2} m r^2 \dot{\theta}^2$$
$$+ m\sin\theta\,\dot{y}\dot{r} + mr\cos\theta\,\dot{y}\dot{\theta} + m\cos\theta\,\dot{r}\dot{x} - mr\sin\theta\,\dot{\theta}\dot{x}$$

$$U = 2(\tfrac{1}{2} k_x x^2) + 2(\tfrac{1}{2} k_y y^2) + \tfrac{1}{2} k r^2$$

$$R = 2(\tfrac{1}{2} c_x \dot{x}^2) + 2(\tfrac{1}{2} c_y \dot{y}^2) + 2(\tfrac{1}{2} c \dot{r}^2)$$

Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial R}{\partial \dot{x}} + \frac{\partial U}{\partial x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = M\ddot{x} + m(-2\sin\theta\,\dot{r}\dot{\theta} - \cos\theta\, r\dot{\theta}^2 + \cos\theta\,\ddot{r} + \ddot{x} - r\sin\theta\,\ddot{\theta})$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial R}{\partial \dot{x}} = 2c_x\dot{x}, \quad \frac{\partial U}{\partial x} = 2k_x x$$

$$(m+M)\ddot{x} + m\cos\Omega t\,\ddot{r} + 2c_x\dot{x} - 2m\Omega\sin\Omega t\,\dot{r} + 2k_x x$$
$$- m\Omega^2\cos\Omega t\, r = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} + \frac{\partial R}{\partial \dot{y}} + \frac{\partial U}{\partial y} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = M\ddot{y} + m(-2\cos\theta\,\dot{r}\dot{\theta} - \sin\theta\, r\dot{\theta}^2 + \sin\theta\,\ddot{r} + \ddot{x} - r\cos\theta\,\ddot{\theta})$$

$$\frac{\partial T}{\partial y} = 0, \quad \frac{\partial R}{\partial \dot{y}} = 2c_y\dot{y}, \quad \frac{\partial U}{\partial y} = 2k_y y$$

$$(m+M)\ddot{y} + m\sin\Omega t\,\ddot{r} + 2c_y\dot{y} + 2m\Omega\cos\Omega t\,\dot{r}$$
$$+2k_y y - m\Omega^2\sin\Omega t\, r = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} + \frac{\partial R}{\partial \dot{r}} + \frac{\partial U}{\partial r} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = m(\ddot{r} + \cos\theta(\dot{y}\dot{\theta} + \ddot{x}) + \sin\theta(-\dot{x}\dot{\theta} + \ddot{y}))$$

$$\frac{\partial T}{\partial r} = m\dot{\theta}(-\sin\theta\,\dot{x} + \cos\theta\,\dot{y} + r\dot{\theta})$$

$$\frac{\partial R}{\partial \dot{r}} = 2c\dot{r} \qquad \frac{\partial U}{\partial r} = kr$$

$$m\ddot{r} + m\cos\Omega t\,\ddot{x} + m\sin\Omega t\,\ddot{y} + 2c\dot{r} + (k - m\Omega^2)r = 0$$

In summary

1) $(m+M)\ddot{x} + m\cos\Omega t\,\ddot{r} + 2c_x\dot{x} - 2m\Omega\sin\Omega t\,\dot{r} + 2k_x x$
$\quad - m\Omega^2\cos\Omega t\, r = 0$

2) $(m+M)\ddot{y} + m\sin\Omega t\,\ddot{r} + 2c_y\dot{y} + 2m\Omega\cos\Omega t\,\dot{r}$
$\quad + 2k_y y - m\Omega^2\sin\Omega t\, r = 0$

3) $m\ddot{r} + m\cos\Omega t\,\ddot{x} + m\sin\Omega t\,\ddot{y} + 2c\dot{r} + (k - m\Omega^2)r = 0$

Not needed but we can write in Matrix form as

$$\begin{bmatrix} m+M & 0 & m\cos\Omega t \\ 0 & m+M & m\sin\Omega t \\ m\cos\Omega t & m\sin\Omega t & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{r} \end{Bmatrix} +$$

$$\left(\begin{bmatrix} 2c_x & 0 & 0 \\ 0 & 2c_y & 0 \\ 0 & 0 & 2c \end{bmatrix} + \begin{bmatrix} 0 & 0 & -2m\Omega\sin\Omega t \\ 0 & 0 & 2m\Omega\cos\Omega t \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} \dot{x} \\ \dot{y} \\ \dot{r} \end{Bmatrix} -$$

$$\left(\begin{bmatrix} 2k_x & 0 & 0 \\ 0 & 2k_y & 0 \\ 0 & 0 & k-m\Omega^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -m\Omega^2\cos\Omega t \\ 0 & 0 & -m\Omega^2\sin\Omega t \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} x \\ y \\ r \end{Bmatrix} = \bar{0}$$

(2) Ignoring the x & y dynamics leads to

$$m\ddot{r} + 2c\dot{r} + (k - m\Omega^2)r = 0$$

$$\Omega > \sqrt{k/m} \sim \text{grows unbounded}$$

we ignore the other equations and just look at "r" dynamics.

(3) if $\Omega = 0$ and we ignore damping

$$\underbrace{\begin{bmatrix} m+M & 0 & 0 \\ 0 & m+M & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{r} \end{Bmatrix}}_{\ddot{\bar{X}}} +$$

$$\begin{bmatrix} 2k_x & 0 & 0 \\ 0 & 2k_y & 0 \\ 0 & 0 & k \end{bmatrix} \underbrace{\begin{Bmatrix} x \\ y \\ r \end{Bmatrix}}_{\bar{X}} = \bar{0}$$

assume $\vec{x} = \bar{X} e^{i\omega t}$, $\ddot{\vec{x}} = \omega^2 \bar{X} e^{i\omega t}$

$$\begin{bmatrix} -\omega^2(m+M)+2k_x & 0 & 0 \\ 0 & -\omega^2(m+M)+2k_y & 0 \\ 0 & 0 & -\omega^2 m + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \vec{0}$$

$$Det = \underset{1}{(-\omega^2(m+M)+2k_x)} \underset{2}{(-\omega^2(m+M)+2k_y)} \underset{3}{(-\omega^2 m+k)} = 0$$

1 $-\omega^2(m+M) + 2k_x = 0 \longrightarrow \omega = \pm\sqrt{\dfrac{2k_x}{(m+M)}}$

2 $-\omega^2(m+M) + 2k_y = 0 \longrightarrow \omega = \pm\sqrt{\dfrac{2k_y}{(m+M)}}$

3 $-\omega^2 m + k = 0 \longrightarrow \omega = \sqrt{k/m}$

These are decoupled oscillators so

$\bar{X}^1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$, $\bar{X}^2 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ $\bar{X}^3 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

$\omega_1 = \sqrt{\dfrac{2k_x}{m+M}}$ $\omega_2 = \sqrt{\dfrac{2k_y}{(m+M)}}$ $\omega_3 = \sqrt{k/m}$

④ if $\cos\Omega t = 1$, $\sin\Omega t = 0$

$$\begin{bmatrix} m+M & 0 & m \\ 0 & m+M & 0 \\ m & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{r} \end{Bmatrix} +$$

$$\left(\begin{bmatrix} 2k_x & 0 & 0 \\ 0 & 2k_y & 0 \\ 0 & 0 & k-m\Omega^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -m\Omega^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} x \\ y \\ r \end{Bmatrix} = \vec{0}$$

$$\vec{x} = \bar{X} e^{i\Omega t} \quad \text{and} \quad \ddot{\vec{x}} = -\Omega^2 \bar{X} e^{i\Omega t}$$

$$\begin{bmatrix} -w^2(m+M) + 2k_x & 0 & -mw^2 - m\Omega^2 \\ 0 & -w^2(m+M) + 2k_y & 0 \\ -w^2 m & 0 & -w^2(m) + k - m\Omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \bar{0}$$

see Mathematica Code

$$w_1 = \sqrt{\frac{2k_y}{m+M}} \qquad \bar{X}_1 = \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

$$w_2 = \sqrt{\frac{-k(m+M) + M(m\Omega^2 - 2k_x) + \sqrt{-8k_x(-m^2 + mM + M^2) + (k(m+M) + M(2k_x - m\Omega^2))^2}}{2(m^2 - mM - M^2)}}$$

$$\bar{X}_2 = \begin{Bmatrix} \dfrac{k(m+M) - M(m\Omega^2 - 2k_x) + \sqrt{-8k_x(-m^2 + mM + M^2) + (k(m+M) + M(2k_x - m\Omega^2))^2}}{4k_x m} \\ 0 \\ 1 \end{Bmatrix}$$

$$w_3 = \sqrt{\frac{k(m+M) - M(m\Omega^2 - 2k_x) + \sqrt{-8k_x(-m^2 + mM + M^2) + (k(m+M) + M(2k_x - m\Omega^2))^2}}{2(m^2 - mM - M^2)}}$$

$$\bar{\bar{X}} = \begin{Bmatrix} \dfrac{+k(m+M) - M(m\Omega^2 - 2k_x) - \sqrt{-8k_x(-m^2 + mM + M^2) + (k(m+M) + M(2k_x - m\Omega^2))^2}}{4k_x m} \\ 0 \\ 1 \end{Bmatrix}$$

---

if $\sin\Omega t = 1 \quad \cos\Omega t = 0$

$$\begin{bmatrix} m+M & 0 & 0 \\ 0 & m+M & m \\ 0 & m & m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{r} \end{Bmatrix} +$$

$$\left(\begin{bmatrix} 2k_x & 0 & 0 \\ 0 & 2k_y & 0 \\ 0 & 0 & k - m\Omega^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -m\Omega^2 \\ 0 & 0 & 0 \end{bmatrix} \right) \begin{Bmatrix} x \\ y \\ r \end{Bmatrix} = \bar{0}$$

$\vec{x} = \bar{\bar{X}} e^{i\Omega t}$ and $\ddot{\vec{x}} = -\Omega^2 \bar{\bar{X}} e^{i\Omega t}$

$$\begin{bmatrix} -w^2(m+M) + 2k_x & 0 & 0 \\ 0 & -w^2(m+M) + 2k_y & -m\Omega^2 - mw^2 \\ 0 & -w^2 m & -w^2(m) + k - m\Omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} =$$

$$w_1 = \sqrt{\frac{2k_y}{m+M}} \qquad \bar{X}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$w_2 = \sqrt{\frac{+k(m+M) + M(-m\Omega^2 + 2k_y) - \sqrt{+8k_y M(m+M)(-k+m\Omega^2) + \left(k(m+M) + M(2k_y - m\Omega^2)\right)^2}}{2M(m+M)}}$$

$$\bar{X}_2 = \begin{Bmatrix} 0 \\ \dfrac{2mM\Omega^2}{-k(m+M) + M(2k_y + m\Omega^2) + \sqrt{8k_y M(m+M)(-k+m\Omega^2) + \left(k(m+M) + M(2k_y - m\Omega^2)\right)^2}} \\ 1 \end{Bmatrix}$$

$$w_3 = \sqrt{\frac{k(m+M) - M(m\Omega^2 - 2k_y) - \sqrt{+8k_y M(m+M)(-k+m\Omega^2) + \left(k(m+M) + M(2k_y - m\Omega^2)\right)^2}}{2M(m+M)}}$$

$$\bar{X} = \begin{Bmatrix} 0 \\ \dfrac{-2mM\Omega^2}{+k(m+M) - M(2k_y + m\Omega^2) + \sqrt{8k_y M(m+M)(-k+m\Omega^2) + \left(k(m+M) + M(2k_y - m\Omega^2)\right)^2}} \\ 1 \end{Bmatrix}$$

In[289]:=

```
Clear["Global *"];
Remove["Global`*"];
```

Case 1

In[291]:=

```
MMM = {{m + M, 0, m}, {0, m + M, 0}, {m, 0, M}}
```

Out[291]=

$\{\{m + M, 0, m\}, \{0, m + M, 0\}, \{m, 0, M\}\}$

In[292]:=

```
MatrixForm[MMM]
```

Out[292]//MatrixForm=

$$\begin{pmatrix} m + M & 0 & m \\ 0 & m + M & 0 \\ m & 0 & M \end{pmatrix}$$

In[293]:=

```
KKK = {{2 * kx, 0, -m * Ω^2}, {0, 2 * ky, 0}, {0, 0, k - m * Ω^2}}
```

Out[293]=

$\{\{2\,kx, 0, -m\,\Omega^2\}, \{0, 2\,ky, 0\}, \{0, 0, k - m\,\Omega^2\}\}$

In[294]:=

```
MatrixForm[KKK]
```

Out[294]//MatrixForm=

$$\begin{pmatrix} 2\,kx & 0 & -m\,\Omega^2 \\ 0 & 2\,ky & 0 \\ 0 & 0 & k - m\,\Omega^2 \end{pmatrix}$$

In[295]:=

```
FullSimplify[{vals, vecs} = Eigensystem[{KKK, MMM}]]
```

Out[295]=

$$\left\{\left\{\frac{2\,ky}{m+M},\ \frac{-k\,m-k\,M-2\,kx\,M+m\,M\,\Omega^2+\sqrt{-8\,kx\left(-m^2+m\,M+M^2\right)\left(k-m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,kx-m\,\Omega^2\right)\right)^2}}{2\left(m^2-m\,M-M^2\right)},\ \frac{k\,m+k\,M+2\,kx\,M-m\,M\,\Omega^2+\sqrt{-8\,kx\left(-m^2+m\,M+M^2\right)\left(k-m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,kx-m\,\Omega^2\right)\right)^2}}{2\left(-m^2+m\,M+M^2\right)}\right\},\right.$$

$$\left\{\{0,1,0\},\ \left\{\frac{k\,(m+M)-M\left(2\,kx+m\,\Omega^2\right)+\sqrt{-8\,kx\left(-m^2+m\,M+M^2\right)\left(k-m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,kx-m\,\Omega^2\right)\right)^2}}{4\,kx\,m},\ 0,\ 1\right\},\right.$$

$$\left.\left.\left\{-\frac{-k\,(m+M)+M\left(2\,kx+m\,\Omega^2\right)+\sqrt{-8\,kx\left(-m^2+m\,M+M^2\right)\left(k-m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,kx-m\,\Omega^2\right)\right)^2}}{4\,kx\,m},\ 0,\ 1\right\}\right\}\right\}$$

Case 2

In[296]:=

```
MMM = {{m + M, 0, 0}, {0, m + M, 0}, {0, m, M}}
```

Out[296]=

$\{\{m+M, 0, 0\}, \{0, m+M, 0\}, \{0, m, M\}\}$

In[297]:=

```
MatrixForm[MMM]
```

Out[297]//MatrixForm=

$$\begin{pmatrix} m+M & 0 & 0 \\ 0 & m+M & 0 \\ 0 & m & M \end{pmatrix}$$

In[298]:=

```
KKK = {{2 * kx, 0, 0}, {0, 2 * ky, -m * Ω^2}, {0, 0, k - m * Ω^2}}
```

Out[298]=

$\left\{\{2\,kx, 0, 0\}, \left\{0, 2\,ky, -m\,\Omega^2\right\}, \left\{0, 0, k-m\,\Omega^2\right\}\right\}$

In[299]:=

```
MatrixForm[KKK]
```

Out[299]//MatrixForm=

$$\begin{pmatrix} 2\,kx & 0 & 0 \\ 0 & 2\,ky & -m\,\Omega^2 \\ 0 & 0 & k-m\,\Omega^2 \end{pmatrix}$$

In[300]:=

```
FullSimplify[{vals, vecs} = Eigensystem[{KKK, MMM}]]
```

Out[300]=

$$\left\{\left\{\frac{2\,kx}{m+M},\ \frac{k\,m+k\,M+2\,ky\,M-m\,M\,\Omega^2-\sqrt{8\,ky\,M\,(m+M)\,\left(-k+m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,ky-m\,\Omega^2\right)\right)^2}}{2\,M\,(m+M)},\right.\right.$$

$$\left.\frac{k\,m+k\,M+2\,ky\,M-m\,M\,\Omega^2+\sqrt{8\,ky\,M\,(m+M)\,\left(-k+m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,ky-m\,\Omega^2\right)\right)^2}}{2\,M\,(m+M)}\right\},$$

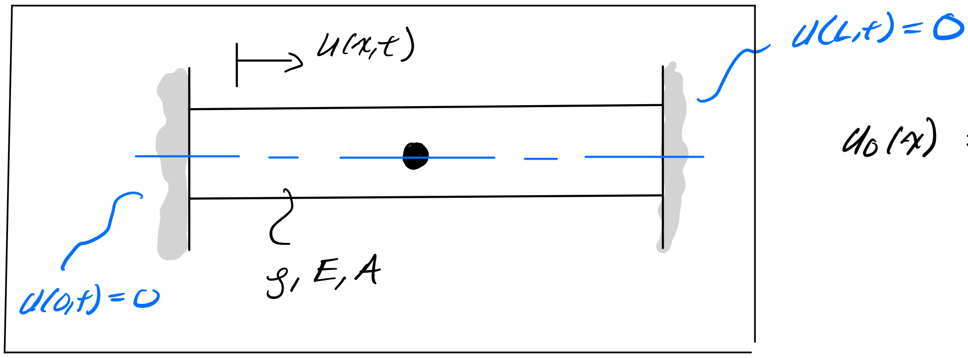
$$\left\{\{1,0,0\},\right.$$

$$\left\{0,\ \frac{2\,m\,M\,\Omega^2}{-k\,(m+M)+M\left(2\,ky+m\,\Omega^2\right)+\sqrt{8\,ky\,M\,(m+M)\,\left(-k+m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,ky-m\,\Omega^2\right)\right)^2}},\ 1\right\},$$

$$\left\{0,\right.$$

$$\left.\left.\left.-\frac{2\,m\,M\,\Omega^2}{k\,(m+M)-M\left(2\,ky+m\,\Omega^2\right)+\sqrt{8\,ky\,M\,(m+M)\,\left(-k+m\,\Omega^2\right)+\left(k\,(m+M)+M\left(2\,ky-m\,\Omega^2\right)\right)^2}},\ 1\right\}\right\}\right\}$$

Homework 4.1



$$u_0(x) = \begin{cases} \varepsilon x, & 0 < x \le L/2 \\ \varepsilon(L-x), & L/2 < x \le L \end{cases}$$

Equation of Motion: $EA\dfrac{\partial^2 u}{\partial x^2} = \rho A \dfrac{\partial^2 u}{\partial t^2}$

Separable Solution: $u(x,t) = U(x)T(t)$

$$E U''(x) T(t) = \rho U(x) \ddot{T}(t) \longrightarrow \frac{E U''(x)}{\rho U(x)} = \frac{\ddot{T}(t)}{T(t)} = -\omega^2$$

① $U''(x) + \dfrac{\omega^2 \rho}{E} U(x) = 0 \longrightarrow U''(x) + \beta U(x) = 0$

$\beta = \omega\sqrt{\rho/E}$

② $\ddot{T}(t) + \omega^2 T(t) = 0$

The solutions are of the form:

$T(t) = C\cos\omega t + S\sin\omega t$, $U(x) = a\sin\beta x + b\cos\beta x$

Evaluate Boundary Conditions

$u(0,t) = U(0)\,T(t) = 0$

$u(L,t) = U(L)\,T(t) = 0$

$U(0) = 0 = a\cancelto{1}{\cos(0)} + b\cancelto{0}{\sin(0)} = A = 0$

$U(L) = 0 = a\cancelto{0}{\cos(\beta L)} + b\sin(\beta L) = B\sin\beta L$

The characteristic equation: $\underline{CE = b\sin\beta L}$

$b = 0$ is the trivial solution where $U(x) = 0$

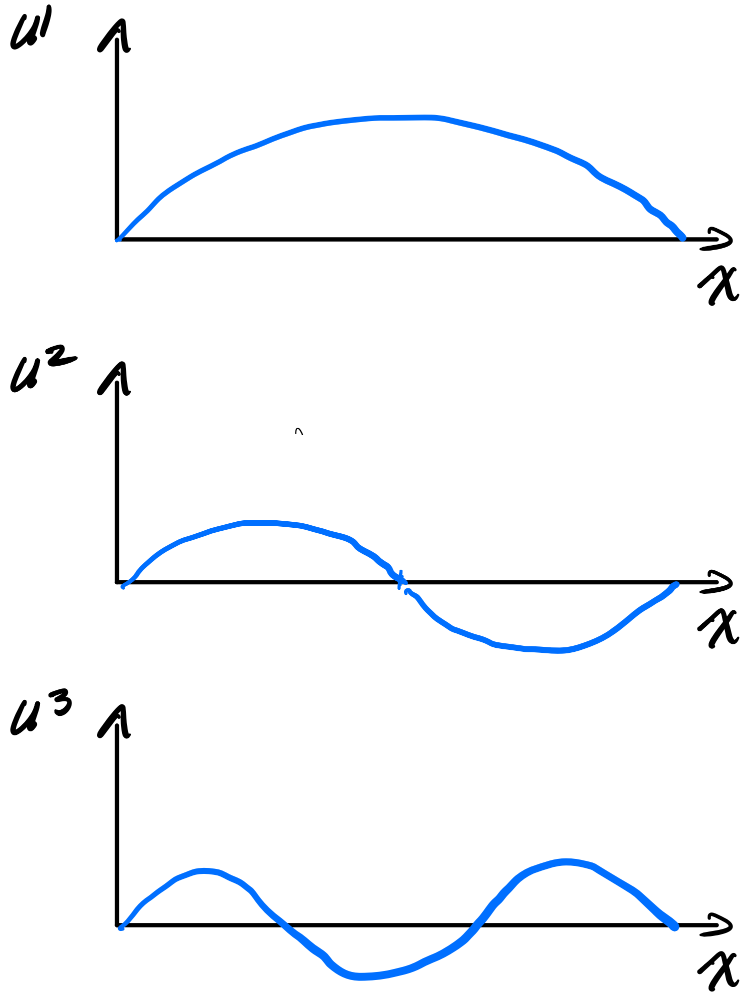
$\sin\beta L = 0 \qquad \beta L = \pi, 2\pi, 3\pi \ldots$

$\beta_n = n\pi/L, \quad n = 1, 2, 3, \ldots \infty$

Now, $\beta = \omega\sqrt{\rho/E} \longrightarrow \beta_n = n\pi/L = \omega_n\sqrt{\rho/E}$

$\omega_n = \sqrt{E/\rho}\,\beta_n = \sqrt{E/\rho}\,(n\pi/L) = n\pi\sqrt{E/\rho L^2}$

$U_n(x) = b_n \sin(n\pi x/L) \sim$ Modal Function



1st 3 mode shapes

Turning attention to initial value problem

$\ddot{T} + w^2 T = 0$ becomes

$\ddot{T}_n + w_n^2 T_n = 0$, $n = 1, 2, 3, \ldots \infty$

and the solution

$T(t) = C \cos wt + S \sin wt$ becomes

$T_n(t) = C_n \cos w_n t + S_n \sin w_n t$

Evaluate IC's

3) $u(x,0) = u_0(x) = \sum_{n=1}^{\infty} U_n(x) C_n$

4) $\frac{\partial u}{\partial t}(x,0) = 0 = \sum_{n=1}^{\infty} w_n U_n(x) S_n$

We will use the formulas

$$C_n = \frac{\int_0^L U_n(x)\, u_0(x,0)\, dx}{\int_0^L U_n(x)^2\, dx}$$

$$S_n = \frac{\int_0^L U_n(x)\, \frac{\partial u_0}{\partial t}(x,0)\, dx}{w_n \int_0^L U_n(x)^2\, dx}$$

Let's evaluate $\int_0^L U_n(x)\, u_0(x,0)\, dx$ since $u_0(x)$ is symmetric

$$\int_0^L U_n(x)\, u_0(x,0)\, dx = \begin{cases} 2\int_0^{L/2} U_n(x)\, u_0(x,0)\, dx & n = \text{odd} \\ 0 & n = \text{even} \end{cases}$$

only look at $n=$ odd

$$\int_0^L U_n(x)\,u_0(x,0)\,dx = 2\int_0^{L/2} u_0(x)\,U_n(x)\,dx$$

$$= 2\int_0^{L/2} \varepsilon x \sin(n\pi x/L)\,dx$$

$$= 2\varepsilon\left\{ \frac{-x\cos n\pi/2}{n\pi/L} + \frac{1}{(n\pi/L)^2}\sin n\pi x/L \right\}\Bigg|_0^{L/2}$$

$$= 2\varepsilon\left\{ \cancel{\frac{-L/2\cos n\pi/2}{n\pi/L}}^{\,\to 0} + \frac{1}{(n\pi/L)^2}\sin n\pi/2 \right\}$$

$n=1, 3, 5, \ldots$

$$= \frac{2\varepsilon L^2}{n^2\pi^2}\sin n\pi/2$$

Now evaluate $\int_0^L U_n(x)^2\,dx = \int_0^L (\sin n\pi x/L)^2\,dx$

$$= L/2$$

$$C_n = \frac{\int_0^L U_n(x)\,u_0(x,0)\,dx}{\int_0^L U_n(x)^2\,dx} = \frac{\begin{cases} 2\int_0^{L/2} U_n(x)\,u_0(x,0)\,dx, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}}{L/2}$$

$$C_n = \begin{cases} \frac{1}{L}\int_0^{L/2} u_n(x)\, u_0(x,0)\, dx, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$C_n = \begin{cases} \frac{2\varepsilon L^2}{n^2\pi^2} \sin n\pi/2, & n = \text{odd} \\ 0, & n = \text{even} \end{cases}$$

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin(n\pi x/L) \cos\left(n\pi \sqrt{E/\rho L^2}\; t\right)$$