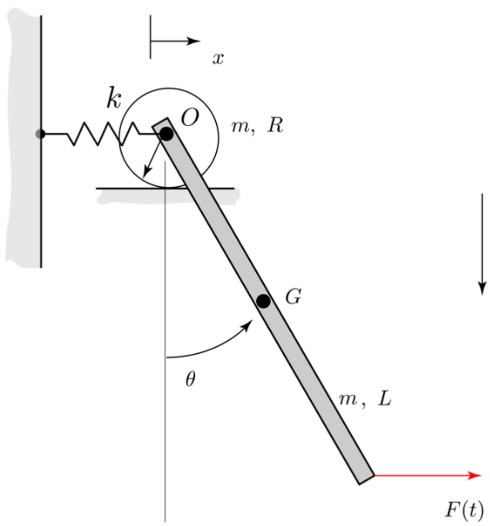
ME 563 - Fall 2020
Homework Problem 6.3

A forcing $F(t) = f_0 \sin\Omega t$ acts at the end of the thin, homogeneous bar of the two-DOF system shown below. The wheel can be modeled as a cylinder with mass m, radius R and rolls without slipping. The response of the system is to be described by the coordinates $x(t)$ and $\theta(t)$. Let $g/L = 2k/m$.



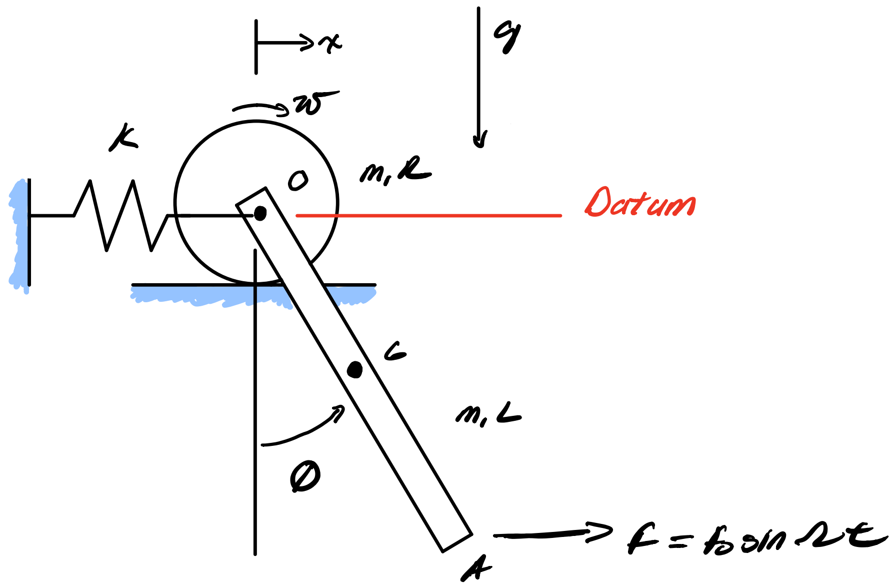
a) Derive the particular solutions $x_p(t)$ and $\theta_p(t)$ for the system.

b) At what values of the temporal frequency Ω does resonance occur in the system?

c) Show that the "shape" of the response is that of the first mode when excited at the first natural frequency, and that the shape is that of the second mode when excited at the second natural frequency.-

d) At what values (if any) of the temporal frequency Ω do anti-resonances occur for $x_p(t)$? For $\theta_p(t)$?

e) Make plots for the amplitudes of $x_p(t)$ and $\theta_p(t)$ vs. the temporal frequency Ω.



Derive Equations of Motion use Lagrange's Methods

The position vectors $\vec{r}_O = x\hat{i}$ and

$$\vec{r}_G = x\hat{i} - \frac{L}{2}\cos\phi\,\hat{j} + \frac{L}{2}\sin\phi\,\hat{i}$$

$$= \left(x + \frac{L}{2}\sin\phi\right)\hat{i} - \frac{L}{2}\cos\phi\,\hat{j}$$

The velocity vector is $\vec{v}_O = \dot{x}\hat{i}$

and $\vec{v}_G = \left(\dot{x} + \dot{\phi}\frac{L}{2}\cos\phi\right)\hat{i} + \dot{\phi}\frac{L}{2}\sin\phi\,\hat{j}$

Now, calculating the kinetic energy

$$T = \frac{1}{2} m\, \vec{v}_G \cdot \vec{v}_G + \frac{1}{2} m\, \vec{v}_O \cdot \vec{v}_O + \frac{1}{2} I_O \omega^2 + \frac{1}{2} I_G \dot{\phi}^2$$

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{x}\dot{\phi} L\cos\phi + \frac{\dot{\phi} L^2}{4}\right) + \frac{1}{2} I_O \omega^2 + \frac{1}{2} I_G \dot{\phi}^2$$

Kinematic $\vec{v}_O = R\omega = \dot{x}$ and $I_O = \frac{1}{2} m R^2$

therefor $\omega = \dot{x}/R$ and $I_G = \frac{1}{12} m L^2$

$$T = \tfrac{1}{2} m\left(\dot{x}^2 + \dot{x}\dot{\theta}L\cos\theta + \frac{\dot{\theta}L^2}{4}\right) + \tfrac{1}{2}\underset{\frac{1}{2}mR^2}{\cancel{I_0}}\,\overset{(\dot{x}/R)^2}{\cancel{\omega}^2} + \tfrac{1}{2}\underset{\frac{1}{12}mL^2}{\cancel{I_G}}\,\dot{\theta}^2$$

$$T = \tfrac{1}{2}\underset{m_{xx}}{\left(\tfrac{3}{2}m\right)}\dot{x}^2 + \underset{m_{x\theta}+m_{\theta x}}{\tfrac{1}{2}m\dot{x}\dot{\theta}L\cos\theta} + \tfrac{1}{2}\underset{m_{\theta\theta}}{\left(\tfrac{1}{3}mL^2\right)}\dot{\theta}^2$$

Now, the mass matrix can be extracted

$$[m] = \begin{bmatrix} \frac{3}{2}m & \frac{mL\cos\theta}{2} \\ \frac{mL\cos\theta}{2} & \frac{mL^2}{3} \end{bmatrix}$$

The potential energy is

$$U = \tfrac{1}{2}kx^2 - mg\tfrac{L}{2}\cos\theta$$

The equilibrium point can be calculated as

$$\frac{\partial U}{\partial x} = kx$$

$$\frac{\partial U}{\partial \theta} = mg\tfrac{L}{2}\sin\theta$$

$$\frac{\partial U}{\partial x_i} = 0 \longrightarrow \begin{cases} kx = 0 \\ mg\frac{L}{2}\sin\theta = 0 \end{cases}$$

$$\begin{cases} x_{eq} = 0 \\ \theta_{eq} = n\pi \end{cases} \qquad n = 0, 1, 2, \ldots \infty$$

Let's look at $n=0$ $(x_{eq}, \theta_{eq}) = (0, 0)$

The stiffness matrix

$$k_{xx} = \frac{\partial^2 U}{\partial x^2} = k$$

$$k_{x\theta} = k_{\theta x} = \frac{\partial^2 U}{\partial x \partial \theta} = 0$$

$$k_{\theta\theta} = \frac{\partial^2 U}{\partial \theta^2} = mg\tfrac{L}{2}\cos\theta$$

$$[m] = \begin{bmatrix} \tfrac{3}{2}m & mL/2 \\ mL/2 & mL^2/3 \end{bmatrix}$$

for $(x_{eq}, \theta_{eq}) = (0,0)$

$$[k] = \begin{bmatrix} k & 0 \\ 0 & mgL/2 \end{bmatrix}$$

Calculating the generalized force

$dW = \bar{F} \cdot d\bar{r}_A$ where

$$\bar{r}_A = (x + L\sin\theta)\,\hat{i} - \tfrac{L}{2}\cos\theta\,\hat{j}$$

$$d\bar{r}_A = (dx + L\cos\theta\, d\theta)\,\hat{i} + \tfrac{L}{2}\sin\theta\, d\theta\,\hat{j}$$

$$\bar{F} = f\,\hat{i}$$

$$dW = f\hat{i} \cdot (dx + L\cos\theta\, d\theta)\,\hat{i}$$

$$= f\,dx + fL\cos\theta\, d\theta$$

$\overset{2}{Q_x}$ $\overset{1}{Q_\theta}$

The final equations of motion

$$\begin{bmatrix} \tfrac{3}{2}m & mL/2 \\ mL/2 & mL^2/3 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & mgL/2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} f \\ fL \end{Bmatrix}$$

Now $f(t) = f_0 \sin \Omega t$ and $g/L = 2k/m$

$$mg/2L = k \rightarrow mgL^2/2L = kL^2$$

$$\begin{bmatrix} \frac{3}{2}m & mL/2 \\ mL/2 & mL^2/3 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & kL^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} f_0 \\ f_0 L \end{Bmatrix} \sin \Omega t$$

It is convient to make $L\theta$ one state

$$\boxed{\begin{bmatrix} \frac{3}{2}m & m/2 \\ m/2 & m/3 \end{bmatrix} \begin{Bmatrix} x \\ L\theta \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} x \\ L\theta \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} f_0 \sin \Omega t}$$

1st calculate Modeshapes and Natural Frequencies

$$\bar{x} = \begin{Bmatrix} x \\ L\theta \end{Bmatrix} = \underline{X}\, e^{j\omega t}$$

Plug into Homogeneous form of governing equation

$$\begin{bmatrix} \frac{3}{2}m & m/2 \\ m/2 & m/3 \end{bmatrix} \begin{Bmatrix} x \\ L\theta \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} x \\ L\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

to yield

$$\left(-\omega^2 \begin{bmatrix} \frac{3}{2}m & m/2 \\ m/2 & m/3 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \right) \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\omega^2 \tfrac{3}{2}m + k & -\omega^2 m/2 \\ -\omega^2 m/2 & -\omega^2 m/3 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

$$\underbrace{\phantom{-\omega^2 \tfrac{3}{2}m + k \quad -\omega^2 m/3 + k}}_{D}$$

$$\det(D) = \left(-\omega^2 \tfrac{3}{2}m + k\right)\left(-\omega^2 m/3 + k\right) - \omega^4 m/4 = 0$$

solving in matlab and only using "+" ω

$$\omega_{1,2} = 0.770\sqrt{k/m} \text{ and } 2.596\sqrt{k/m}$$

Use 1st equation to calculate mode shap

$$\left(-\omega^2 \tfrac{3}{2}m + k\right) X_1 + \left(\omega^2 m/2\right) X_2 = 0$$

$$\left(-\omega^2 \tfrac{3}{2}m + k\right) X_1 = -\omega^2 m/2 \, X_2$$

$$\frac{X_2}{X_1} = \frac{-\omega^2 \tfrac{3}{2}m + k}{-\omega^2 m/2}$$

Natural Frequencies and Modeshapes

$\omega_1 = 0.770\sqrt{k/m}$ $\qquad \dfrac{X_2}{X_1} = 0.37$

$\omega_2 = 2.596\sqrt{k/m}$ $\qquad \dfrac{X_2}{X_1} = -2.70$

Now, back to forced problem using Complex Exponential

$$\boxed{\begin{bmatrix} \tfrac{3}{2}m & \tfrac{m}{2} \\ \tfrac{m}{2} & \tfrac{m}{3} \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ L\ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} x \\ L\theta \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} f_0 \sin \Omega t}$$

$$[m]\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{F}_0\, Imag(e^{i\Omega t})$$

assume $\vec{x} = \vec{X} e^{i\Omega t}$ $\quad f(t) = \vec{F}_0 e^{i\Omega t}$

Plug into matrix equation

$$\left[-\Omega^2 [m] + [k] \right] \vec{X} = \vec{F}_0$$

$$\vec{X} = \left[-\Omega^2 [m] + [k] \right]^{-1} \vec{F}_0$$

$$\vec{X} = \begin{bmatrix} -\Omega^2 \tfrac{3}{2}m + k & -\Omega^2 \tfrac{m}{2} \\ -\Omega^2 \tfrac{m}{2} & -\Omega^2 \tfrac{m}{3} + k \end{bmatrix}^{-1} \begin{Bmatrix} f_0 \\ f_0 \end{Bmatrix}$$

$$\vec{X} = \frac{1}{\Delta} \begin{bmatrix} -\Omega^2 \tfrac{m}{3} + k & +\Omega^2 \tfrac{m}{2} \\ \Omega^2 \tfrac{m}{2} & -\Omega^2 \tfrac{3}{2}m + k \end{bmatrix} \begin{Bmatrix} f_0 \\ f_0 \end{Bmatrix}$$

$$\Delta = (-\Omega^2 \tfrac{3}{2}m + k)(-\Omega^2 \tfrac{m}{3} + K) + \Omega^4 \tfrac{m}{4}$$

$$\bar{X} = \frac{1}{\Delta}\begin{Bmatrix} \Omega^2 m/6 + k \\ -\Omega^2 m + k \end{Bmatrix} f_0$$

See Matlab Code

Now at $\Omega = \omega_1 = 0.770\sqrt{k/m}$

$$\frac{X_2}{X_1} = \frac{-\omega_1^2 m + k}{-\omega^2 m/6 + k} = 0.37$$

Now at $\Omega = \omega_2 = 2.546\sqrt{k/m}$

$$\frac{X_2}{X_1} = \frac{-\omega_2^2 m + k}{-\omega_2^2 m/6 + k} = -2.70$$

Same as modeshape

Note: that anti-resonance can for $X_2 = 0$ when $\Omega = \sqrt{k/m}$

Now for hand plots consider

$$X_1/f_0 = (\Omega^2 m/6 + k)/\Delta$$
$$X_2/f_0 = (-\Omega^2 m + k)/\Delta$$
$$\Delta = (-\Omega^2 \tfrac{3}{2} m + k)(-\Omega^2 m/3 + k) + \Omega^4 m/4$$

① $\Omega \to 0$

$\Delta = k^2$ $\quad X_1/f_0 \to 1/k$, $X_2/f_0 \to 1/k$

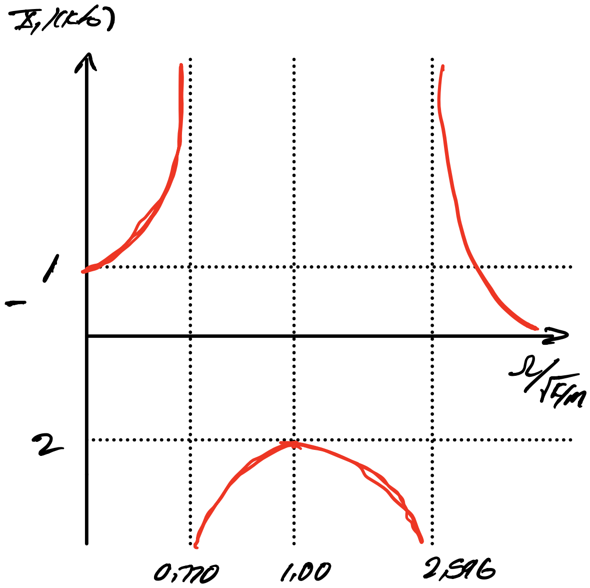
② $\Omega \to \infty$

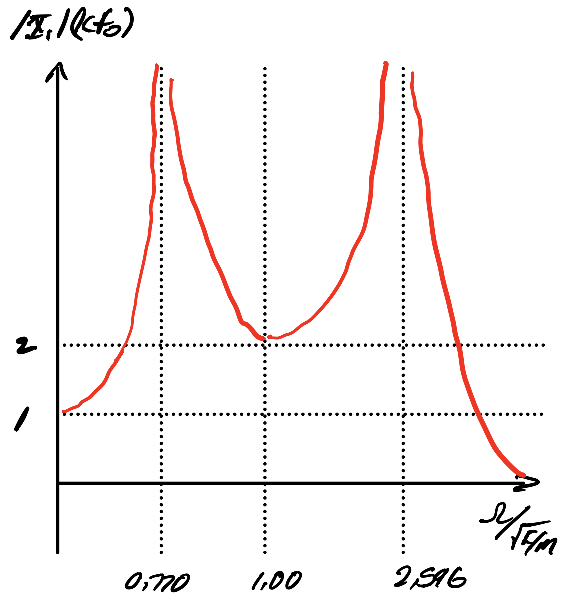
$\Delta \to \infty$ $\quad X_1/f_0 \to 0$, $X_2/f_0 \to 0$

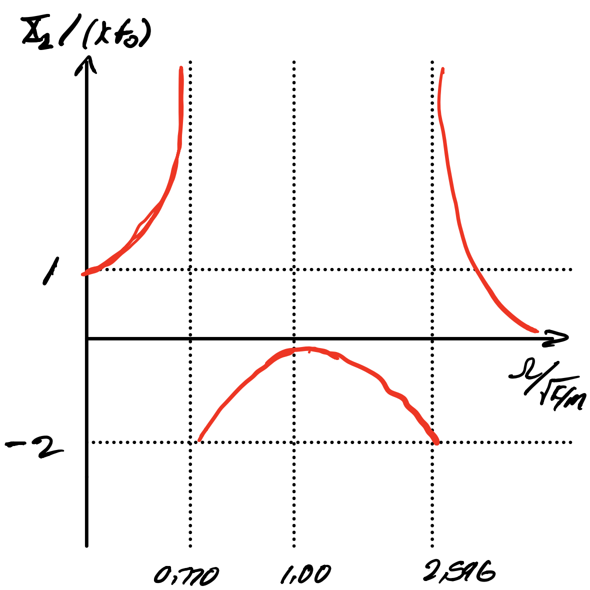
③ $\Omega = \omega_1$ or ω_2 $\quad X_1/f_0 \to \infty$, $X_2/f_0 \to \infty$

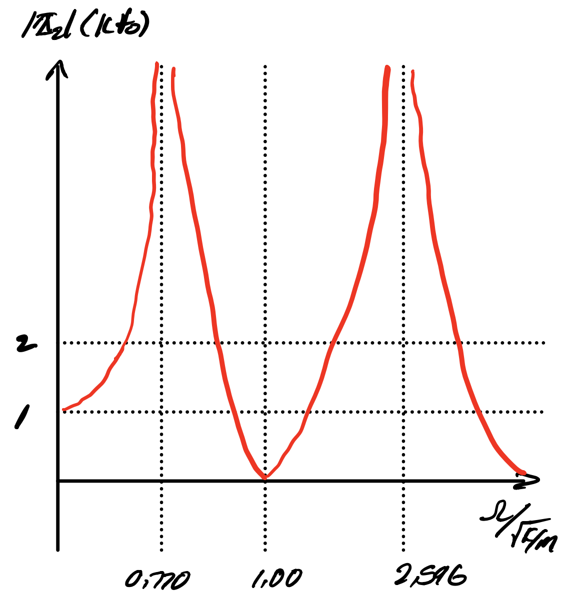
④ $\Omega = \sqrt{k/m}$ $\quad X_1/f_0 = -2f_0/k$, $X_2/f_0 = 0$

Ω	X_1	X_2
0	$1/k$	$1/k$
∞	0	0
$\omega_1 = 0.770\sqrt{k/m}$	∞	∞
$\omega_2 = 2.596\sqrt{k/m}$	∞	∞
$\sqrt{k/m}$	$-2f_0/k$	0


$X_1/(kf_0)$
1
2
$\Omega/\sqrt{k/m}$
0,770
1,00
2,596


$|X_1|/(kf_0)$
2
1
$\Omega/\sqrt{k/m}$
0,770
1,00
2,596


$X_2/(kf_0)$
1
-2
$\Omega/\sqrt{k/m}$
0,770
1,00
2,596


$|X_2|/(kf_0)$
2
1
$\Omega/\sqrt{k/m}$
0,770
1,00
2,596

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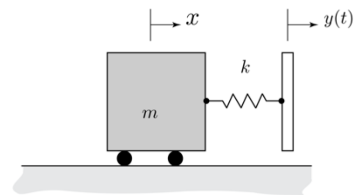
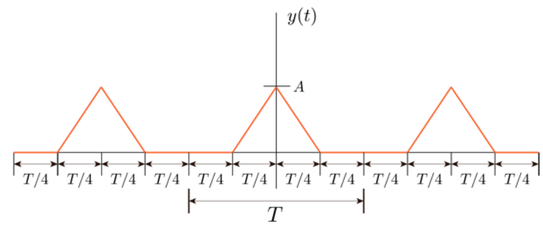
__Homework Problem 7.1__



The undamped, single-DOF system above is given a base excitation of $y(t)$. The base motion $y(t)$ is T-periodic in time, as shown above.



a) Determine the Fourier series of $y(t)$.

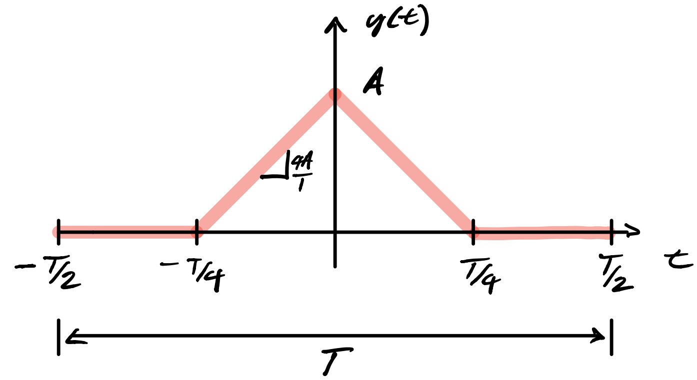
b) Use Matlab to make a plot of $y(t)$ vs. t/T for your Fourier series. Use a sufficient number of terms in your Fourier series when making the plot to insure that the series has converged.

c) Derive the equation of motion for the system.

d) Determine the particular solution for the response, $x_P(t)$.

e) Use Matlab to make a plot of $x_P(t)$ vs. t/T for your Fourier series corresponding to ~~$T = 0.87\,T_n$,~~ where $T_n = 2\pi\,(m/k)^{1/2}$ is the natural period of oscillation for the system. Use the same number of terms in your response $x_P(t)$ as you did in your Fourier series for $y(t)$ above.

$T_n/T = 0.87$



We can write $y(t)$ as a piecewise function

$$y(t) = \begin{cases} 0; & -T/2 < t < -T/4 \\ +\frac{4A}{T}t + A; & -T/4 \le t \le 0 \\ -\frac{4At}{T} + A; & 0 \le t \le T/4 \\ 0; & T/4 \le t \le T/2 \end{cases}$$

This is a symmetric function

Write as a Fourier Series

$$y(t) = y_0 + \sum_{j=1}^{\infty} \left(y_{cj} \cos j\Omega t + y_{sj} \sin j\Omega t \right)$$

Calculate the Fourier Coefficients

$$y_0 = 1/T \int_{-T/2}^{T/2} y(t)\,dt$$

$$y_0 = 1/T \int_{-T/4}^{0} \left(\frac{4A}{T} t + A\right) dt + 1/T \int_{0}^{T/4} \left(-\frac{4A}{T} t + A\right) dt$$

$$= 2/T \int_{0}^{T/4} \left(-\frac{4A}{T} t + A\right) dt$$

$$= 2/T \left(\frac{-2At^2}{T} + At\right)\Bigg|_0^{T/4} = \frac{2}{T}\left(\cancelto{-\frac{2AT}{16}}{\frac{-2AT^2}{16T}} + \cancelto{\frac{4AT}{16}}{\frac{AT}{4}}\right) = \frac{2}{T}\left(\frac{2AT}{16}\right) = \frac{A}{4}$$

$$y_{cj} = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\cos(j\Omega t)\,dt$$ since $y(t)$ is symmetric

$$= \frac{4}{T}\int_0^{T/2} y(t)\cos(j\Omega t)\,dt$$

$$= \frac{4}{T}\int_0^{T/4}\left(-\frac{4A}{T}t + A\right)\cos j\Omega t\,dt$$

$$= \frac{4A}{T}\int_0^{T/4}\left(\frac{-4t}{T} + 1\right)\cos j\Omega t\,dt$$

$$= \frac{4A}{T}\left\{\int_0^{T/4} -\frac{4t}{T}\cos j\Omega t\,dt + \int_0^{T/2}\cos j\Omega t\,dt\right\}$$

$$= \frac{4A}{T}\left\{\left(-\frac{4t}{T}\left(\frac{1}{j\Omega}\right)\sin j\Omega t - \frac{4}{T}\left(\frac{1}{j^2\Omega^2}\right)\cos j\Omega t\right)\Bigg|_0^{T/4} + \frac{1}{j\Omega}\sin j\Omega t\Bigg|_0^{T/4}\right\}$$ where $\Omega = 2\pi/T$

$$= \frac{4A}{j^2\pi^2}\left(1 - \cos j\pi/2\right)$$

$$y_{sj} = \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(j\Omega t)\,dt$$ since $y(t)$ is symmetric

$$= 0$$

The total solution $y(t) = A\left(\frac{1}{4} + \frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left(1-\cos j\pi/2\right)\cos 2\pi j t/T\right)$

See Matlab Code

Write EOM

$T = \frac{1}{2} m \dot{x}^2 \qquad U = \frac{1}{2} k (x-y)^2$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) + \frac{\partial U}{\partial x} = 0$$

$$m\ddot{x} + kx = ky$$

$$m\ddot{x} + kx = kA\left(\frac{1}{4} + \frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left(1-\cos j\pi/2\right)\cos 2\pi j t/T\right)$$

$$\ddot{x} + \omega_n^2 x = \omega_n^2 A\left(\frac{1}{4} + \frac{4}{\pi^2}\sum_{j=1}^{\infty}\frac{1}{j^2}\left(1-\cos j\pi/2\right)\cos(j\Omega t)\right)$$

with $\Omega = 2\pi/T$

The particular solution is

$$x_p(t) = x_0 + \sum_{j=1}^{\infty} x_{cj}(t)$$

Now, evaluating $x_0 = C$

$$\underset{0}{\cancel{\ddot{x}_0}} + \omega_n^2 x_0 = \omega_n^2 \frac{A}{4} \qquad x_0 = A/4$$

Now, evaluating $x_{cj}(t) = B_j \cos(j\Omega t)$

$$\ddot{x}_{cj} + \omega_n^2 x_{cj} = \omega_n^2 A\left(\frac{4}{\pi^2}(1/j^2)(1-\cos j\pi/2)\cos(j\Omega t)\right)$$

$$-(j\Omega^2) B_j \cancel{\cos j\Omega t} + \omega_n^2 B_j \cancel{\cos j\Omega t}$$
$$= \omega_n^2 A\left(\frac{4}{\pi^2}(1/j^2)(1-\cos j\pi/2)\cancel{\cos(j\Omega t)}\right)$$

$$B_j = \frac{\omega_n^2}{\omega_n^2 - (j\Omega)^2}\ \frac{4A}{\pi^2}\left(1/j^2\right)\left(1-\cos j\pi/2\right)$$

Recall $\qquad \Omega = 2\pi/T, \qquad \omega_n = 2\pi/T_n$

$$B_j = \frac{1}{1-\left(j\Omega/\omega_n\right)^2}\,\frac{4A}{\pi^2}\left(\frac{1}{j^2}\right)\left(1-\cos j\pi/2\right)$$

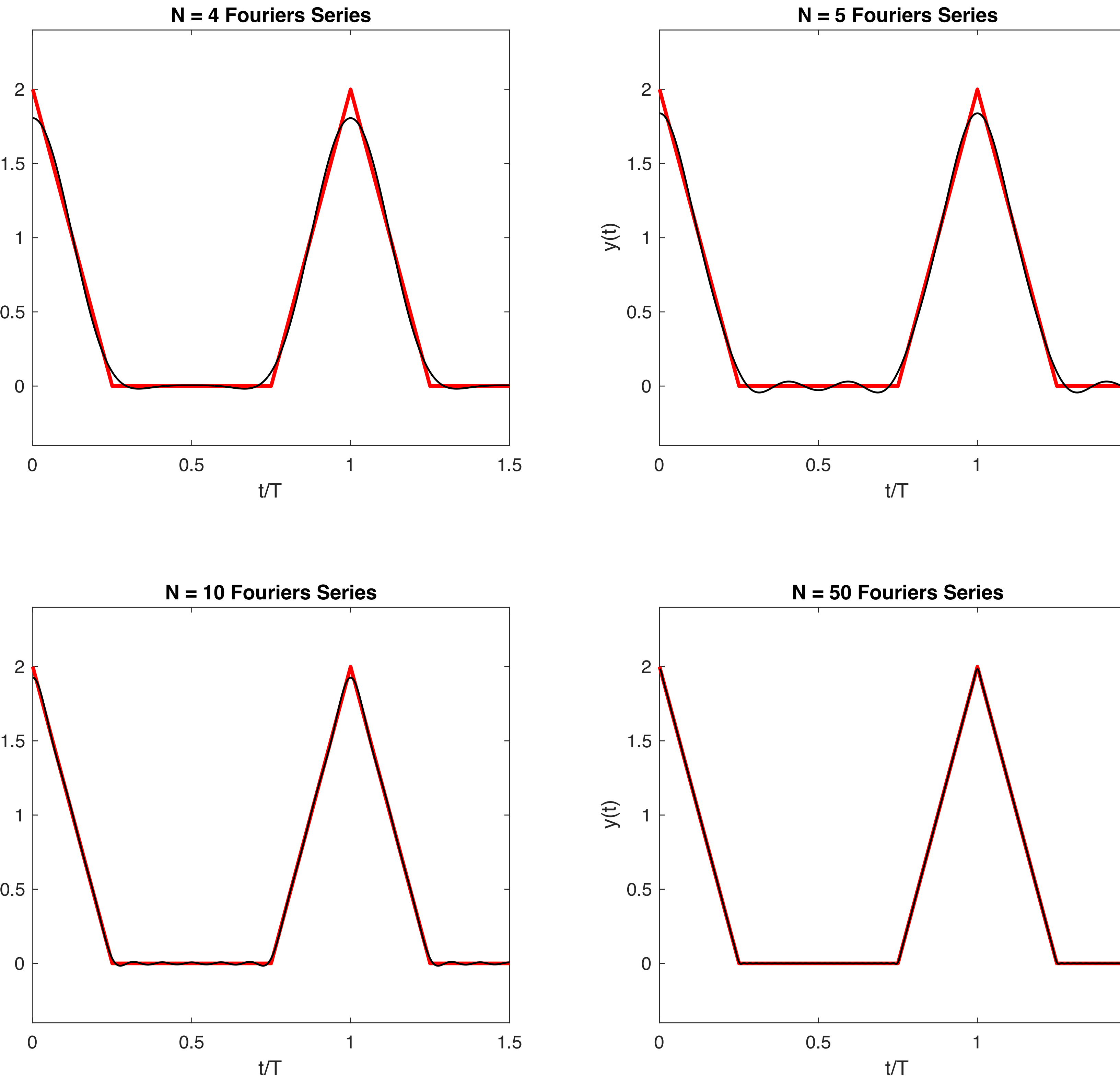
$$B_j = \frac{1}{1-\left(jT/T_n\right)^2}\,\frac{4A}{\pi^2}\left(\frac{1}{j^2}\right)\left(1-\cos j\pi/2\right)$$

The total solution

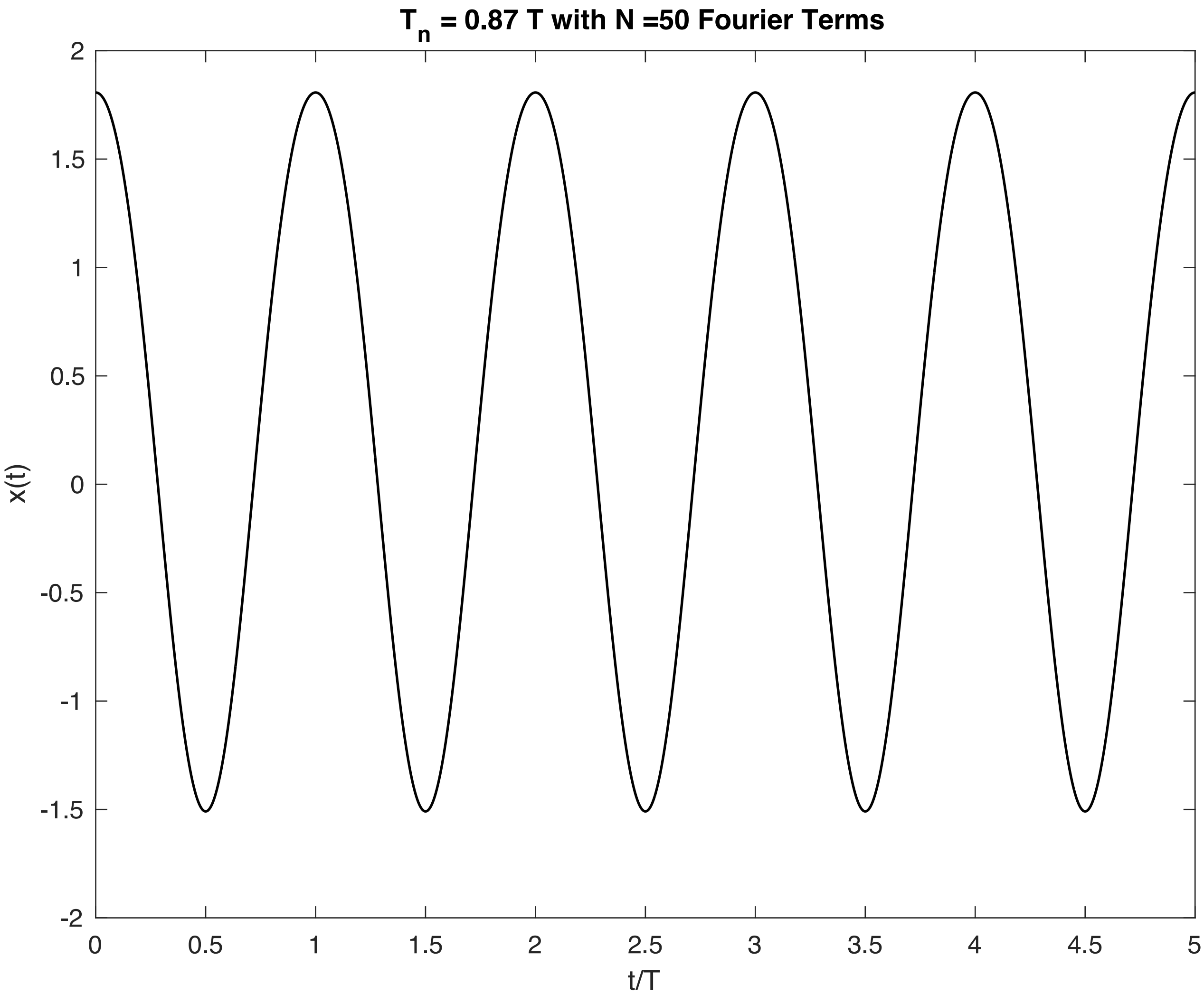
$$\boxed{x_p(t) = \frac{A}{4} + \frac{4A}{\pi^2}\sum_{j=1}^{\infty}\left(\frac{1}{1-\left(jT_n/T\right)^2}\right)\left(\frac{1}{j^2}\right)\left(1-\cos\frac{j\pi}{2}\right)\cos 2\pi j t/T}$$

```
clc
clear all
close all

syms T

N  = 100; %Number of Fourier Terms


T  = 1;
A  = 2;
F  = [ A 0   0   0     A  0     0 ];
tp  = [ 0 T/4 T/2 3*T/4 T 5*T/4 3/2*T  ];

%%
% Fourier Series

Nv = [4 5 10 50 ];
t  = linspace(0,3/2*T,1000);
for k = 1: length(Nv)
    N = Nv(k);
    y0    = A/4;
    yt    = y0;
    ysum  = 0;
    Omega = 2*pi/T;
    Omegav = 0;
    ycj    = 0;
    for j= 1:N
        ycj(j)      = 4*A/pi^2*(1/j^2)*(1-cos(j*pi/2));
        ysum        = ysum + ycj(j)*cos(2*pi*j*t/T);
    end
    yt = yt+ ysum;
    subplot(2,2,k)
    line(tp,F,'linewidth',2,'color','r')
    line(t,yt,'linewidth',1,'color','k')

    xlabel('t/T')
    ylabel('y(t)')
    title(['N = ', num2str(N), ' Fouriers Series'])
    axis([-inf inf -0.2*A 1.2*A])
    box on
end
%%
% Particular Solution
t  = linspace(0,5*T,1000);
N     = 50;
x0    = A/4;
xsum  = 0;
Omega = 2*pi/T;
xcj   = 0;
T     = 1;
Tn    = 0.87*T;
for j= 1:N
    xcj(j) = (4*A/pi^2)*(1/j^2)*(1-cos(j*pi/2))*(1/(1 - (j*Tn/T)^2));
    xsum   = xsum + xcj(j)*cos(2*pi*j*t/T);
end
xt = x0 + xsum;
figure
```

```
line(t/T,xt/A,'linewidth',1,'color','k')
xlabel('t/T')
ylabel('x(t)')
title(['T_n = ' ,num2str(Tn), ' T with N =',num2str(N), ' Fourier Terms'])
box on
```



N = 4 Fouriers Series
N = 5 Fouriers Series
N = 10 Fouriers Series
N = 50 Fouriers Series
y(t)
t/T
0
0.5
1
1.5
2



T_n = 0.87 T with N =50 Fourier Terms
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
x(t)
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t/T

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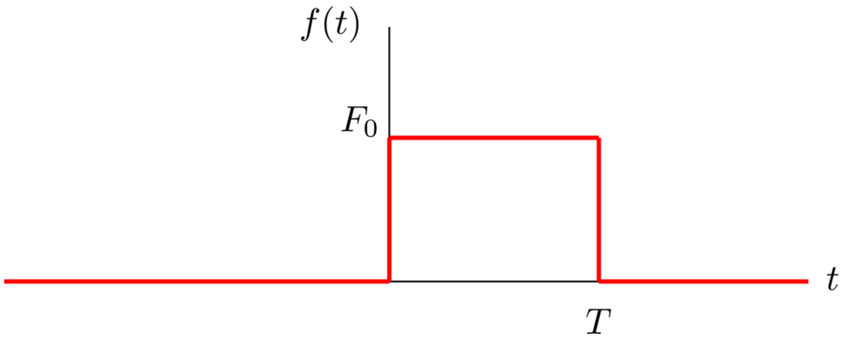
Homework Problem 7.2



Consider a damped, single-DOF system with an equation of motion of:
$m\ddot{x} + c\dot{x} + kx = F(t)$

It was shown in lecture that the convolution integral for the undamped case ($\zeta = 0$) with zero initial conditions
$x(0) = \dot{x}(0) = 0$

that the convolution integral could be written as:

$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

where the impulse response function was given by:

$$h(t-\tau) = \frac{1}{m\omega_n} \sin \omega_n (t-\tau).$$

a) Show that the convolution integral solution for zero initial conditions for the critically-damped case ($\zeta = 1$) can be written in the same general form:

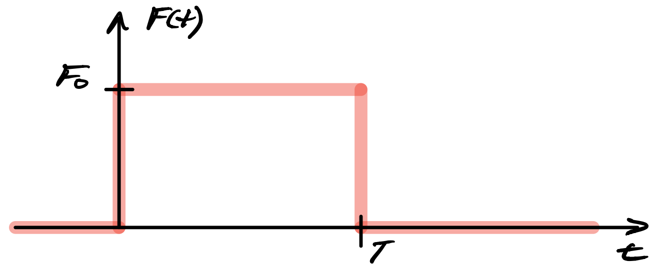
$$x(t) = \int_0^t h(t-\tau) f(\tau) d\tau$$

What is the impulse response function $h(t-\tau)$ in this case? Feel free to start your derivation using the general form of the convolution integral found in the lecturebook in terms of the fundamental solutions $u(t)$ and $v(t)$.

b) Use the convolution integral derived above in a) to determine the *response of the system* to the forcing shown above, where $T = 0.50\ T_n$, where $T_n = 2\pi\ (m / k)^{1/2}$ is the natural period of oscillation for the system,

$m\ddot{x} + c\dot{x} + kx = F(t)$

$x(0) = \dot{x}(0) = 0$



Primary Derivation

$$x(t) = au(t) + bv(t) + \frac{1}{m}\int_0^t \frac{u(\tau)v(t) - u(t)v(\tau)}{u(\tau)\dot{v}(\tau) - \dot{u}(\tau)v(\tau)} f(\tau)d\tau$$

if $\zeta = 1$ and there are 0 initial conditions

$a = b = 0$

$u(t) = e^{-\zeta\omega_n t}$ and $v(t) = te^{-\zeta\omega_n t}$

Now,

$\dot{u}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}$ and

$\dot{v}(t) = e^{-\zeta\omega_n t} - \zeta\omega_n t e^{-\zeta\omega_n t}$

let's examine the numerator

$$u(\tau)v(t) - u(t)v(\tau) = e^{-\zeta\omega_n \tau} t e^{-\zeta\omega_n t} - e^{-\zeta\omega_n \tau} \tau e^{-\zeta\omega_n \tau}$$
$$= (t-\tau)e^{-\zeta\omega_n(t+\tau)}$$

Now look at denominator

$$u(\tau)\dot{v}(\tau) - \dot{u}(\tau)v(\tau) = -e^{-\zeta\omega_n \tau}(e^{-\zeta\omega_n \tau} - \zeta\omega_n \tau e^{-\zeta\omega_n \tau})$$
$$+ \zeta\omega_n e^{-\zeta\omega_n \tau}(\tau e^{-\zeta\omega_n \tau}) = e^{-2\zeta\omega_n \tau}$$

Plug into the original equation

$$x(t) = \frac{1}{m}\int_0^t \frac{(t-\tau)e^{-\zeta\omega_n(t+\tau)}}{e^{-\zeta\omega_n \tau}} f(\tau)\,d\tau$$

$$x(t) = \frac{1}{m}\int_0^t \underbrace{(t-\tau)e^{-\zeta\omega_n(t-\tau)}}_{h(t-\tau)} f(\tau)\,d\tau$$

$$h(t-\tau) = \frac{1}{m}(t-\tau)e^{-\zeta\omega_n(t-\tau)}$$

Secondary Derivation (You only need 1)

We can use the response when $\zeta = 1$ with $x_0 = 0$ and $v_0 = 1/m$

$$x(t) = (\cancel{x_0}^{\to 0} + (\cancel{v_0}^{\to 1} + \omega_n \cancel{x_0}^{\to 0})t)e^{-\omega_n t}$$

$$x(t) = \frac{1}{m} t e^{-\omega_n t} = h(t)$$

and $h(t-\tau) = \frac{1}{m}(t-\tau)e^{-\zeta\omega_n(t-\tau)}$

Derive Response

$$F(t) = \begin{cases} F_0 ; & t < T \\ 0 & t > T \end{cases}$$

for $t < T$

$$x(t) = \frac{1}{m}\int_0^t (t-\tau)e^{-\zeta\omega_n(t-\tau)} f(\tau)\,d\tau$$

$$= \frac{F_0}{m}\int_0^t (t-\tau)e^{-\zeta\omega_n(t-\tau)}\,d\tau$$

Let $z \triangleq t - \tau \qquad dz = -d\tau$

$\tau \triangleq t - z$

Transform limits

$\tau = 0 \rightarrow z = t$

$\tau = t \longrightarrow z = 0$

$\therefore$

$$x(t) = \frac{F_0}{m}\int_0^t (t-\tau)e^{-\zeta\omega_n(t-\tau)}\,d\tau$$

$$= -\frac{F_0}{m}\int_t^0 z e^{-\zeta\omega_n z}\,dz$$

$$= -\frac{F_0}{m}\left\{ -\frac{z}{\zeta\omega_n}e^{-\zeta\omega_n z}\Big|_t^0 + \frac{1}{\zeta\omega_n}\int_t^0 e^{-\zeta\omega_n z}\,dz \right\}$$

$$= -\frac{F_0}{m}\left\{ -\frac{z}{\zeta\omega_n}e^{-\zeta\omega_n z}\Big|_t^0 - \frac{1}{(\zeta\omega_n)^2}e^{-\zeta\omega_n z}\Big|_t^0 \right.$$

$$\boxed{x(t) = \frac{F_0}{m}\left\{ \frac{-t}{\zeta\omega_n}e^{-\zeta\omega_n t} + \frac{1}{(\zeta\omega_n)^2}\left(1 - e^{-\zeta\omega_n t}\right) \right\}}$$

for $t > T$

$$x(t) = \frac{1}{m}\int_0^t (t-\tau)e^{-\zeta\omega_n(t-\tau)} f(\tau)\,d\tau$$

$$= \frac{1}{m}\int_0^T (t-\tau)e^{-\zeta\omega_n(t-\tau)} \cancel{f(\tau)}^{F_0}\,d\tau \; +$$

$$= \frac{1}{m}\int_T^t (t-\tau)e^{-\zeta\omega_n(t-\tau)} \cancel{f(\tau)}^{0}\,d\tau$$

$$= \frac{F_0}{m}\int_T^t (t-\tau)e^{-\zeta\omega_n(t-\tau)}\,d\tau$$

Let $z \triangleq t-\tau \qquad dz = -d\tau$

$\tau \triangleq t - z$

Transform limits

$\tau = 0 \longrightarrow z = t$

$\tau = T \longrightarrow z = t-T$

$$x(t) = \frac{F_0}{m}\int_T^t (t-\tau)e^{-\zeta\omega_n(t-\tau)}\,d\tau$$

$$= -\frac{F_0}{m}\int_t^{t-T} z e^{-\zeta\omega_n z}\,dz$$

$$\boxed{x(t) = -\frac{F_0}{m}\left\{\frac{1}{\zeta\omega_n}\left((t-T)e^{-\zeta\omega_n(t-T)} - t e^{-\zeta\omega_n t}\right) + \frac{1}{(\zeta\omega_n)^2}\left(e^{-\zeta\omega_n(t-T)} - e^{-\zeta\omega_n t}\right)\right\}}$$

In summary,

for $t < T$

$$x(t) = \frac{F_0}{m}\left\{ \frac{-t}{\zeta\omega_n} e^{-\zeta\omega_n t} + \frac{1}{(\zeta\omega_n)^2}\left(1 - e^{-\zeta\omega_n t}\right) \right\}$$

for $t > T$

$$x(t) = -\frac{F_0}{m}\left\{ \frac{1}{\zeta\omega_n}\left((t-T)\, e^{-\zeta\omega_n (t-T)} - t e^{-\zeta\omega_n t}\right) + \frac{1}{(\zeta\omega_n)^2}\left(e^{-\zeta\omega_n (t-T)} - e^{-\zeta\omega_n t}\right) \right\}$$

Now, $\omega_n = 2\pi/T_n$ $\qquad \frac{T}{T_n} = 0.5$

for $t < T$

$$x(t) = \frac{F_0}{m}\left\{ \frac{-t T_n}{-\zeta 2\pi} e^{-\zeta \frac{2\pi t}{T_n}} + \frac{T_n^2}{(\zeta 2\pi)^2} (1 - e^{-\zeta \frac{2\pi t}{T_n}} \right\}$$

for $t > T$

$$x(t) = -\frac{F_0}{m}\left\{ \frac{T_n}{\zeta 2\pi}\left((t-T)\, e^{-\zeta 2\pi\left(\frac{t}{T_n} - 0.5\right)} - t e^{-\zeta(2\pi t/T_n)}\right) + \frac{T_n^2}{(\zeta 2\pi)^2}\left(e^{-\zeta 2\pi(t/T_n - 0.5)} - e^{-\zeta 2\pi t/T_n}\right) \right\}$$