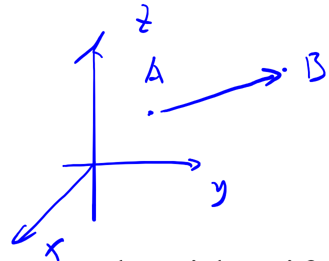
VECTOR DEFINITIONS; DIRECTION COSINES; DIRECTION ANGLES

Learning Objectives

1). To determine and understand the differences between *position vectors*, *unit vectors* and *force vectors*.

2). To determine the *direction cosines* and *direction angles*.

4). To do an *engineering estimate* of these quantities.



Definitions

Position Vector ($\overline{r_{AB}}$): a vector used to identify the **position** of a point in space relative to a reference point.

a vector

$$\boxed{\overline{r_{AB}} = (x, y, z)_B - (x, y, z)_A}$$

from A to B

Unit Vector ($\overline{u_{AB}}$): a *dimensionless* vector of *unit magnitude* that is often used to describe the **direction** of a vector of interest.

$$\boxed{\overline{u_{AB}} = \frac{\overline{r_{AB}}}{|\overline{r_{AB}}|}}$$

a vector

magnitude of $\overline{r_{AB}}$

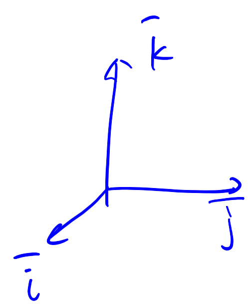
(a scalar, not a vector)

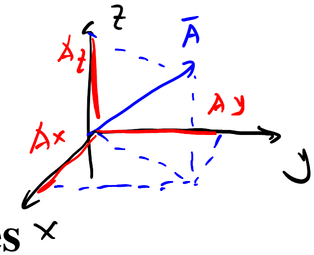
Force Vector ($\overline{F_{AB}}$): a vector used to represent the **magnitude** and **direction** of a force.

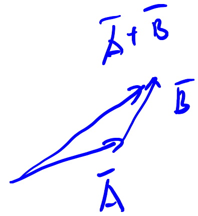
$$\boxed{\overline{F_{AB}} = F_{AB}\ \overline{u_{AB}}}$$

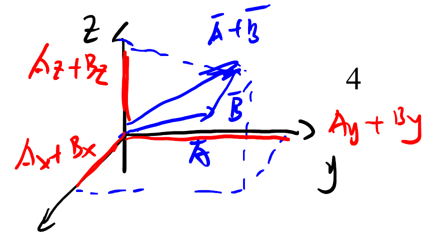
magnitude of Force

Let $\overline{i}$ = unit vector pointing in the x-direction
$\overline{j}$ = unit vector pointing in the y-direction
$\overline{k}$ = unit vector pointing in the z-direction









Consequences

(i) The vector $\overline{\mathrm{A}}$ can be written in terms of its "components" and the unit vectors $\overline{\mathrm{i}}, \overline{\mathrm{j}}$ and $\overline{\mathrm{k}}$ as (see figure):

$$\overline{\mathrm{A}} = \mathrm{A_x}\overline{\mathrm{i}} + \mathrm{A_y}\overline{\mathrm{j}} + \mathrm{A_z}\overline{\mathrm{k}}$$

(ii) The sum of two vectors is accomplished by adding together the respective components:

$$\overline{\mathrm{A}} + \overline{\mathrm{B}} = (\mathrm{A_x}\overline{\mathrm{i}} + \mathrm{A_y}\overline{\mathrm{j}} + \mathrm{A_z}\overline{\mathrm{k}}) + (\mathrm{B_x}\overline{\mathrm{i}} + \mathrm{B_y}\overline{\mathrm{j}} + \mathrm{B_z}\overline{\mathrm{k}})$$

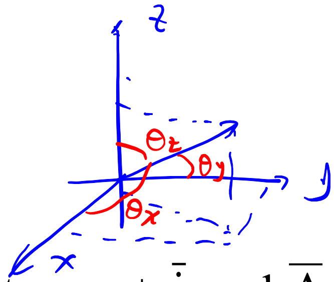
$$= (\mathrm{A_x} + \mathrm{B_x})\overline{\mathrm{i}} + (\mathrm{A_y} + \mathrm{B_y})\overline{\mathrm{j}} + (\mathrm{A_z} + \mathrm{B_z})\overline{\mathrm{k}}$$

(iii) The *magnitude* (length) of $\overline{\mathrm{A}}$ is given by (using Pythagorean Theorem):

$$|\overline{\mathrm{A}}| = \sqrt{\mathrm{A_x^2} + \mathrm{A_y^2} + \mathrm{A_z^2}}$$

(iv) A unit vector u pointing in the same direction as $\overline{\mathrm{A}}$ is given by dividing $\overline{\mathrm{A}}$ its magnitude:

$$\overline{\mathrm{u}} = \frac{\overline{\mathrm{A}}}{|\overline{\mathrm{A}}|} = \underbrace{\frac{\mathrm{A_x}}{|\overline{\mathrm{A}}|}}_{u_x}\overline{\mathrm{i}} + \underbrace{\frac{\mathrm{A_y}}{|\overline{\mathrm{A}}|}}_{u_y}\overline{\mathrm{j}} + \underbrace{\frac{\mathrm{A_z}}{|\overline{\mathrm{A}}|}}_{u_z}\overline{\mathrm{k}}$$



Direction Angles

$\theta_x = \cos^{-1}(A_x / |\bar{A}|)$ = angle between + $\bar{i}$ and $\bar{A}$ vectors, measured in the $\bar{i} - \bar{A}$ plane

α u_x

$\theta_y = \cos^{-1}(A_y / |\bar{A}|)$ = angle between + $\bar{j}$ and $\bar{A}$ vectors, measured in the $\bar{j} - \bar{A}$ plane

β u_y

$\theta_z = \cos^{-1}(A_z / |\bar{A}|)$ = angle between + $\bar{k}$ and $\bar{A}$ vectors, measured in the $\bar{k} - \bar{A}$ plane

γ u_z

Direction Cosines

$\cos\theta_x = (A_x / |\bar{A}|) = u_x$ $\rightarrow$ $A_x = |\bar{A}| \cos\theta_x$

$\cos\theta_y = (A_y / |\bar{A}|) = u_y$ $\Rightarrow$ $\bar{u} = \cos\theta_x\,\bar{i} + \cos\theta_y\,\bar{j} + \cos\theta_z\,\bar{k}$

$\cos\theta_z = (A_z / |\bar{A}|) = u_z$

Note: $\sqrt{(\cos\theta_x)^2 + (\cos\theta_y)^2 + (\cos\theta_z)^2} = 1$

<u>Consequences</u>

$\alpha \to \theta_x$
$\beta \to \theta_y$
$\gamma \to \theta_z$

(i) Rearranging we can see:

$$A_x = |\overline{A}| \cos\alpha$$
$$A_y = |\overline{A}| \cos\beta$$
$$A_z = |\overline{A}| \cos\gamma$$

(ii) Therefore we can write

$$\begin{aligned} \overline{A} &= A_x\,\overline{i} + A_y\,\overline{i} + A_z\,\overline{k} \\ &= |\overline{A}| \cos\alpha\,\overline{i} + |\overline{A}| \cos\beta\,\overline{j} + |\overline{A}| \cos\gamma\,\overline{k} \\ &= |\overline{A}| \underbrace{(\cos\alpha\,\overline{i} + \cos\beta\,\overline{j} + \cos\gamma\,\overline{k})}_{\overline{u}} \end{aligned}$$

(iii) Also, from before,

$$\overline{u} = \frac{\overline{A}}{|\overline{A}|} = \underline{\cos\alpha\,\overline{i} + \cos\beta\,\overline{j} + \cos\gamma\,\overline{k}}$$

(iv) If $A_x > 0$, then $0 < \alpha < 90^\circ$

If $A_x < 0$, then $90^\circ < \alpha < 180^\circ$

<u>Final Remark</u>

Practice making the three-dimensional vectors! Show direction angles in figures. Visualization is the first step in understanding.

Position/Unit/Force Vectors

Example 1

Given: Rod OC has two forces applied at ring C as shown. Angle θ below is given as 53.13°.

Find:

a) Determine the position, unit and force vectors for force $\overline{F}_{CB}$ (i.e., F_2).

b) Determine the unit and force vectors for force $\overline{F}_{CA}$ (i.e., F_1).

c) For force $\overline{F}_{CB}$, approximate the direction angles, then calculate these angles.

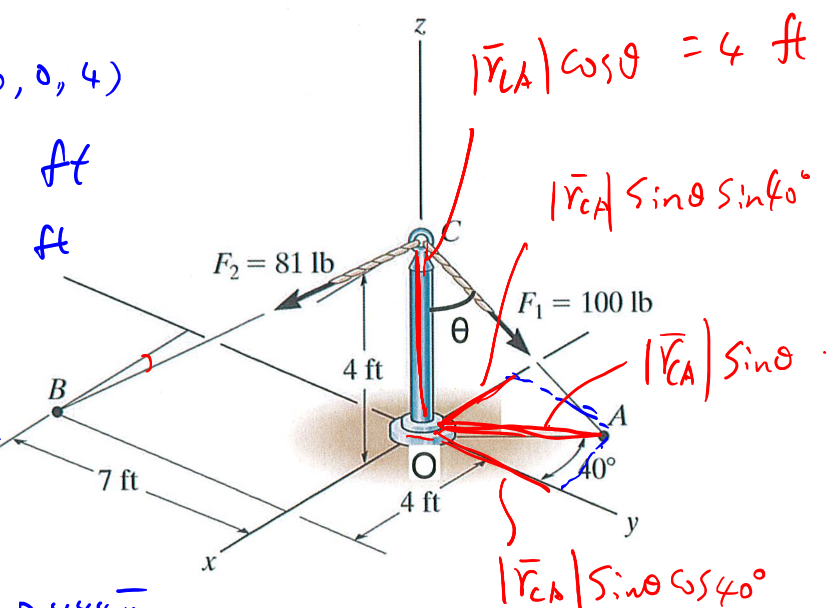
d) For force $\overline{F}_{CB}$, compute the direction cosines.

a) $\overline{r}_{CB} = \overline{B} - \overline{C}$

$= (4, -7, 0) - (0, 0, 4)$

$= (4, -7, -4)$ ft

$= 4\,\bar{i} - 7\,\bar{j} - 4\bar{k}$ ft



$\overline{u}_{CB} = \dfrac{\overline{r}_{CB}}{|\overline{r}_{CB}|} = \dfrac{4\bar{i} - 7\bar{j} - 4\bar{k}}{\sqrt{4^2 + (-7)^2 + (-4)^2}}$

$= \frac{4}{9}\bar{i} - \frac{7}{9}\bar{j} - \frac{4}{9}\bar{k}$

$= 0.444\,\bar{i} - 0.778\,\bar{j} - 0.444\,\bar{k}$

$\overline{F}_{CB} = F_{CB}\overline{u}_{CB} = 81\,(0.444\bar{i} - 0.778\bar{j} - 0.444\bar{k})$

$= 36.0\,\bar{i} - 63.0\,\bar{j} - 36.0\,\bar{k}$ lbs b). $\bar{r}_{CA} = -|\bar{r}_{CA}| \sin\theta \sin 40^\circ \bar{i} + |\bar{r}_{CA}| \sin\theta \cos 40^\circ \bar{j} - |\bar{r}_{CA}| \cos\theta \bar{k}$

$$\bar{u}_{CA} = \frac{\bar{r}_{CA}}{|\bar{r}_{CA}|} = -\sin\theta \sin 40^\circ \bar{i} + \sin\theta \cos 40^\circ \bar{j} - \cos\theta \bar{k}$$

$$= -0.514 \bar{i} + 0.613 \bar{j} - 0.600 \bar{k}$$

$$\bar{F}_{CA} = F_{CA} \bar{u}_{CA} = -51.4 \bar{i} + 61.3 \bar{j} - 60.0 \bar{k} \quad lbs$$

c) $\theta_x \approx 60^\circ$, $\theta_y \approx 150^\circ$ $\theta_z \approx 120^\circ$

$$\theta_x = \cos^{-1}\left[\frac{(\bar{r}_{CB})_x}{|\bar{r}_{CB}|}\right] = \cos^{-1}\left[(\bar{u}_{CB})_x\right] = 63.6^\circ$$

$$\theta_y = \cos^{-1}\left[(\bar{u}_{CB})_y\right] = 141^\circ$$

$$\theta_z = \cos^{-1}\left[(\bar{u}_{CB})_z\right] = 116^\circ$$

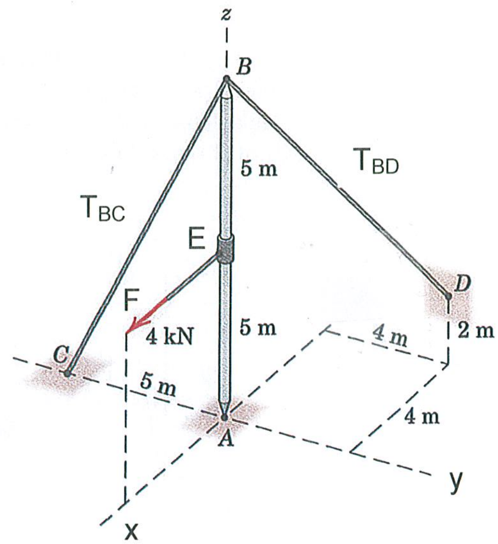
d) $\cos\theta_x = (\bar{u}_{CB})_x = 0.444$

$\cos\theta_y = (\bar{u}_{CB})_y = -0.778$

$\cos\theta_z = (\bar{u}_{CB})_z = -0.444$

Position/Unit/Force Vectors

Example 2



Given: Vertical mast AB supports a 4kN force and is held in static equilibrium by two fixed cables T_{BC} and T_{BD}. The tensions in these cables are T_{BC} = 4.47 kN and T_{BD} is 4.90 kN

Find:

a) Determine the force vector $\overline{T}_{BC}$ and estimate the direction angles of this vector.

b) Determine the force vector $\overline{T}_{BD}$ and calculate the direction angles of this vector.

c) Determine the force vector $\overline{F}_{EF}$.

a) $\overline{r}_{BC} = 0\bar{i} - 5\bar{j} - 10\bar{k}$ m

$\Rightarrow \bar{u}_{BC} = \dfrac{0\bar{i} - 5\bar{j} - 10\bar{k}}{\sqrt{(-5)^2 + (-10)^2}} = 0\bar{i} - 0.447\bar{j} - 0.894\bar{k}$

$\Rightarrow \overline{T}_{BC} = T_{BC}\,\bar{u}_{BC} = 0\bar{i} - 2.00\bar{j} - 4.00\bar{k}$ kN

$\angle BCE \approx 30° \Rightarrow \theta_z \approx 180° - 30° = 50°$

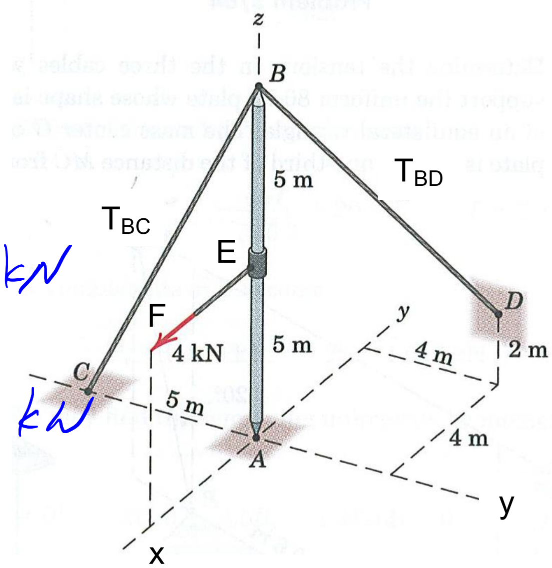
$\theta_x = 90°$ ($\overline{T}_{BC}$ is in y-z plane)

$\theta_y \approx 30° + 90° = 120°$

b) $\bar{T}_{BD} = T_{BD} \, \bar{u}_{BD}$

$$= 4.90 \left(\frac{-4\bar{i} + 4\bar{j} - 8\bar{k}}{\sqrt{(-4)^2 + 4^2 + (-8)^2}} \right) \text{ kN}$$

$$= -2.00\,\bar{i} + 2.00\,\bar{j} - 4.00\,\bar{k} \text{ kN}$$

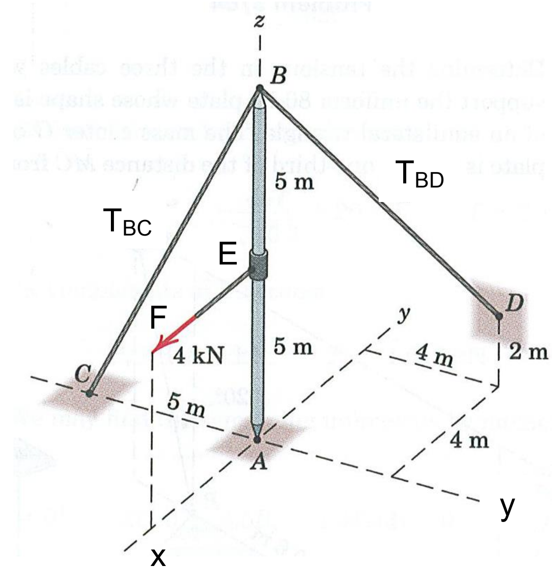


$$\theta_x = \cos^{-1}(-0.408) = 114^\circ$$

$$\theta_y = \cos^{-1}(0.408) = 65.9^\circ$$

c) $\bar{F}_{EF} = F_{EF} \, \bar{u}_{EF} = 4 \left(\frac{4i}{4} \right) = 4\bar{i}$ kN

$$\theta_x = 0^\circ \qquad \theta_y = 90^\circ \qquad \theta_z = 90^\circ$$



z
B
5 m
T_BD
T_BC
E
F
4 kN
5 m
C
D
2 m
y
4 m
5 m
4 m
A
y
x

Position/Unit/Force Vectors
Group Quiz

Group #: ____________ Group Members: 1) ______________________________
(Present Only)

Date: ______________ Period: _________ 2) ______________________________

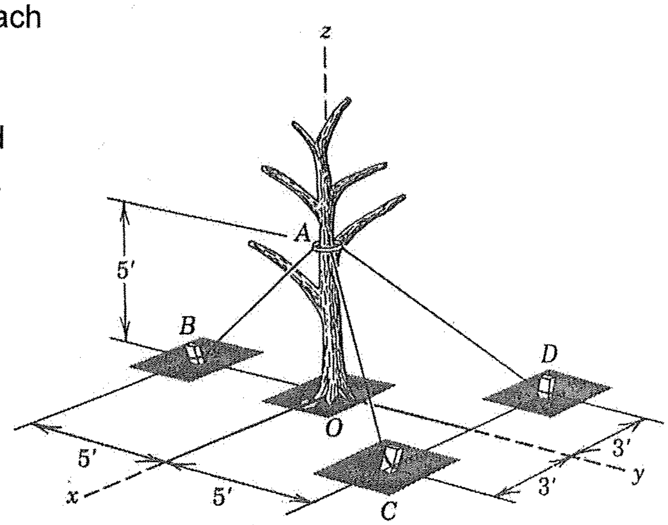
3) ______________________________

4) ______________________________

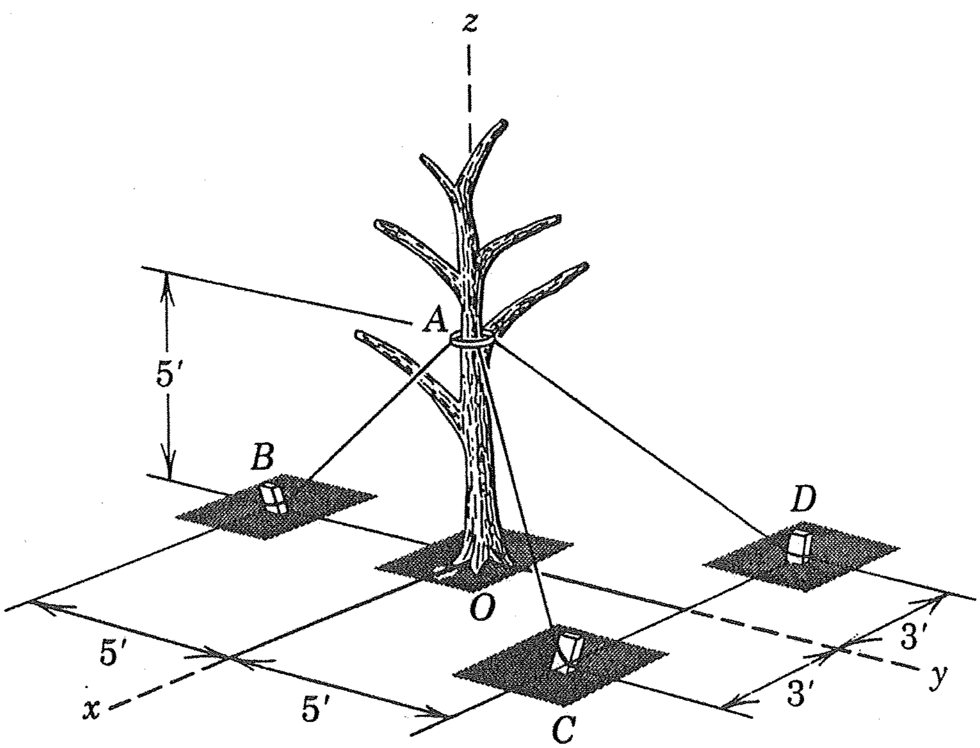
Given: A tree is steadied by three cables as shown. Each cable carries a tension of 1.5 kN.

Find:

a) Determine the position vector, unit vector and force vector for each cable acting on the tree.
b) Estimate the direction angles of cable AB.
c) Calculate the direction angles and direction cosines of cable AB.



Solution:

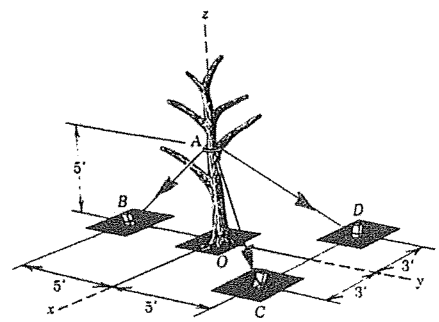

z
A
5′
B
D
O
5′
3′
y
x
5′
C
3′

ME 270 - Basic Mechanics I - Group Quiz

Your Name: SOLUTION Group Members: 1) ______ 2) ______

Date: ______ Period: ______ 3) ______ 4) ______

Given: A tree is steadied by three cables as shown. Each cable carries a tension of 1.5 kN.



Find: (a) Determine the position vector, unit vector and force vector for each cable acting on the tree.
(b) Estimate the direction angles of cable AB.
(c) Calculate the direction angles and direction cosines of cable AB.

Solution: (a)

$$\bar{r}_{AB} = (0,-5,0)_B - (0,0,5)_A = \boxed{-5\bar{j} - 5\bar{k} \text{ ft}}$$

$$|\bar{r}_{AB}| = \left[(-5)^2 + (-5)^2\right]^{1/2} = \sqrt{50} = 7.07 \text{ ft}$$

$$\bar{u}_{AB} = \frac{\bar{r}_{AB}}{|\bar{r}_{AB}|} = -\frac{5}{\sqrt{50}}\bar{j} - \frac{5}{\sqrt{50}}\bar{k} = \boxed{-0.707\bar{j} - 0.707\bar{k}}$$

Unitless

$$\bar{F}_{AB} = F_{AB}\,\bar{u}_{AB} = 1.5\left[-\frac{5}{\sqrt{50}}\bar{j} - \frac{5}{\sqrt{50}}\bar{k}\right]$$

$$\boxed{\bar{F}_{AB} = -1.06\bar{j} - 1.06\bar{k} \text{ kN}}$$

(b) ESTIMATE: $\boxed{\alpha \approx 90^\circ \quad \beta \approx 135^\circ \quad \gamma \approx 135^\circ}$

(c) <u>DIRECTION ANGLES</u> <u>DIRECTION COSINES</u>

$\alpha = \cos^{-1}\left(\frac{0}{\sqrt{50}}\right) = \boxed{90^\circ}$ $\cos\alpha = \boxed{0}$

$\beta = \cos^{-1}\left(\frac{-5}{\sqrt{50}}\right) = \boxed{135^\circ}$ $\cos\beta = \frac{-5}{\sqrt{50}} = \boxed{-0.707}$

$\gamma = \cos^{-1}\left(\frac{-5}{\sqrt{50}}\right) = \boxed{135^\circ}$ $\cos\gamma = \frac{-5}{\sqrt{50}} = \boxed{-0.707}$

(a) $\boxed{\overline{r_{AC}} = 3\bar{i} + 5\bar{j} - 5\bar{k} \text{ ft}}$ $|\overline{r_{AC}}| = \sqrt{59} = 7.68 \text{ ft}$

$\overline{u_{AC}} = \frac{3}{\sqrt{59}}\bar{i} + \frac{5}{\sqrt{59}}\bar{j} - \frac{5}{\sqrt{59}}\bar{k} = \boxed{0.391\bar{i} + 0.651\bar{j} - 0.651\bar{k}}$

$\overline{F_{AC}} = F_{AC}\,\overline{u_{AC}} = \boxed{0.586\bar{i} + 0.976\bar{j} - 0.976\bar{k} \text{ kN}}$

$\overline{r_{AD}} = -3\bar{i} + 5\bar{j} - 5\bar{k}$ $|\overline{r_{AD}}| = \sqrt{59} \quad 7.68 \text{ ft}$

$\overline{u_{AD}} = \frac{-3\bar{i}}{\sqrt{59}} + \frac{5\bar{j}}{\sqrt{59}} - \frac{5}{\sqrt{59}}\bar{k} = \boxed{-0.391\bar{i} + 0.651\bar{j} - 0.651\bar{k}}$

$\overline{F_{AD}} = F_{AD}\,\overline{u_{AD}} = \boxed{-0.586\bar{i} + 0.976\bar{j} - 0.976\bar{k} \text{ kN}}$