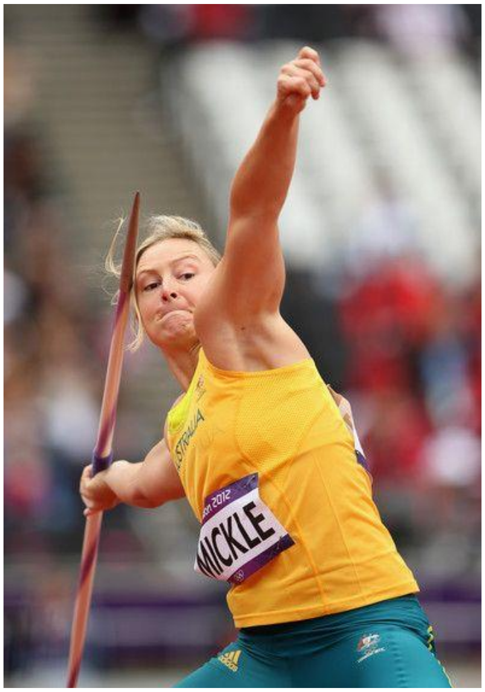


Hardware Store Science Curriculum

Kinetic Motion & Projectile Motion

ICP Maker-STEM Content Document

Key Terms

Acceleration – $a = \Delta v/t$
Elastic Potential Energy –energy stored as a result of deformation of an elastic object $PE_{elastic} = 0.5kx^2$
Free-fall motion – any **motion** of a body where gravity is the only force acting upon it $y = 0.5gt^2$
Gravitational Potential Energy – the energy that an object possesses by virtue of its position relative to others $PE_{gravity} = mgh$
Kinematics – describes the motion objects without considering the forces that caused the motion.
Velocity – $v = d/t$

Lesson 1 – Kinetic Energy: We will begin by reviewing two-dimensional motion and deconstructing all motion into forces acting with horizontal and vertical components. *(ICP.1.1, 1.4, 2.1)*

Lesson 2 – Characteristics of Projectile Motion: Next, we will define projectile motion and describe this motion as the result of independent forces acting perpendicular to one another. *(ICP.1.1, 1.4, 2.1)*

Lesson 3 – Ball Launcher Investigation: Armed with and understanding of projectile motion we will compare calculated launch angle and applied force to actual target striking values obtained during experimentation. *(ICP.4.1-4.4; POE-2)*

Lesson 4 – Projectile Motion Analysis: Next, we will revisit horizontal and vertical projectile motion components as a means of establishing mathematical rules for determining vertical displacement, horizontal displacement and time of flight measurements. *(ICP.1.1, 1.4, 2.1)*

Lesson 5 – Mechanical Energy: Finally, we will learn about kinematic equations. We will focus our attention on motion in one dimension when velocity or acceleration is constant. We will end with a discussion of the four motion variables: displacement, velocity, acceleration and time. *(ICP.1.1, 1.4, 2.1)*

Module 7 Guiding Question

How does potential and kinetic energy affect a projectile's motion?

Projectiles and Conservation

An excellent way to learn about kinetic energy and its interconversion with gravitational potential energy is by studying the motion of **projectiles**. A projectile is any object that has its trajectory influenced by gravity. These can be such simple things as a football that has been kicked, a baseball that has been thrown or a ball that has been shot out of a cannon. Trebuchets and catapults are two iconic projectile systems that allow one to observe the characteristics of projectile motion without references being made to non-gravitational forces. Once a trebuchet or catapult has been activated the projectile is clearly under the influence of the mechanical energy of the launching system or gravity.

We can then apply the concepts of potential and kinetic energy to build a more complete free-body diagram analysis of this motion, as described by Newton's Laws of Motion – in particular Newton's 2nd Law of Motion. A key assumption of the analysis employed in this Module will be the neglect of frictional forces that can slow down the projectile as it moves through the surrounding gas or liquid. Developing an understanding of non-frictional forces acting on an object will be helpful in our understanding of the forces responsible for the motion characterized by Newton's 2nd Law. Once this is mastered a more advanced discussion of projectile motion can occur.

In its most simplified analysis, projectile motion is the motion of an object thrown (or projected) into the air. In this situation the only forces acting on the object once it is motion will be gravity. The object is called a projectile, and its path is called the trajectory. We have been predominantly concerned with the motion of objects in one-dimension, a simple horizontal or vertical movement. Granted, we described that motion using x and y components but the object itself was analyzed in on one dimension at a time. In this module we will consider two-dimensional motion, the motion of a projectile such as a football or other object traveling both horizontally and vertically.

The most important thing for us to remember is that motion along perpendicular axes are independent. Because of this we can analyze each direction separately. The key will be for us to analyze two-dimensional projectile motion by breaking it into motion along a horizontal and vertical axis. This choice of axes is the most sensible, because acceleration due to gravity is vertical. Provided there is no air resistance there will be no acceleration (more appropriately deceleration) along the horizontal axis. We have learned to call the horizontal axis the x-axis and the vertical axis the y-axis. When discussing displacement, the magnitudes of these vectors are x and y and we then used the notation A_x and A_y to designate each component.

As in every analysis of motion we will describe projectile motion with velocity, acceleration, and displacement. When doing so we must find their components along the x and y-axes and assume all forces except gravity are negligible. The y components of acceleration will always be $A_y = g = 9.8m/s^2$ while the x-axis acceleration will be zero ($A_x = g$). This means that both accelerations are constant and kinematic equations can be used

Lesson 1: Objectives

- **Derive the equation for kinetic energy using pendulum motion.**

Content

- **Pendulum Motion**
  - **Period of oscillation**
  - **Gravitational potential energy**
  - **Kinetic energy**
- **Dimensional analysis**

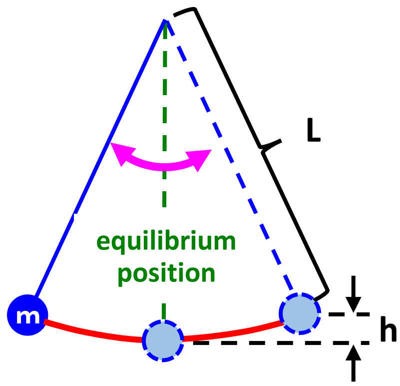
Lesson 1: Kinetic Energy

Kinetic energy is the energy that an object has due to its motion. Everyday examples of kinetic energy are the energy in a moving semi-truck or in a football player running with the ball. During Module 6 we introduced kinetic energy when discussing the total mechanical energy of a system. During this lesson we will learn more about kinetic energy and how it can be calculated.

We will begin by looking at a pendulum, an excellent example of an object with kinetic energy. A pendulum is a device with a fixed weight hung by a wire from a set point, so that it can swing freely back-and-forth. One example of a pendulum is in an old-fashion clock, where the steady swing of the pendulum back-and-forth provides the tick-tock sound and keeps the clock in perfect time. Other examples include a circus performer on a trapeze, a wrecking ball, and a metronome used by musicians to help with timing.




metronome



Consider the diagram of a pendulum's motion, where at equilibrium (i.e. rest) the ball with mass m points straight down. If the mass is pulled to the left and then let go, it then oscillates left and right through this equilibrium position. When the mass is at its far right and far left positions, it is a distance h higher than at the center equilibrium position. As we have already learned, the potential energy change when a mass m is raised by a distance h in a gravitational field is $PE = mgh$. The potential energy in the pendulum varies in magnitude between a minimum ($PE = 0$) at the equilibrium point and a maximum ($PE = mgh$) at the leftmost and rightmost points.

Velocity and speed were discussion in the Speed Velocity Acceleration Module.

We can analyze the kinetic energy associated with the pendulum's motion from left to right. First off, we know that velocity is a measure of an object's motion, and that the pendulum is kinetic energy is related to its velocity as the pendulum goes through its oscillations. We know that velocity is the distance traveled in a specific period of time. In order to accurately determine the velocity of the pendulum mass at various locations, the period of oscillation needs to be long enough to make time and displacement measurements. A pendulum's period of oscillation (T) can be expressed as

$$T = 2\pi\sqrt{\frac{L}{g}}$$

where L is the length of the wire supporting the weight and g is the gravitational constant that on earth is $9.8\mathrm{m/s^2}$ or $32\mathrm{ft/s^2}$ in English units. Notice the period of oscillation does not depend upon the mass m or on the initial displacement of the mass – only on the length L of the wire. The period of oscillation for various wire lengths are:

$L = 1\ ft$	$T = 1.1\ sec$
$L = 6\ ft$	$T = 2.7\ sec$
$L = 40\ ft$	$T = 7.0\ sec$ (height of atrium in Purdue's Forney Hall)
$L = 1250\ ft$	$T = 39.2\ sec$ (height of the Empire State Building)

By definition, an oscillation is when the pendulum goes to the left and then the right. If the time between the *tic* and the *toc* is supposed to be one second, then $1sec = T = 2\pi\sqrt{(L/g)}$, which when solved gives $L = 5.1ft$. This is why the pendulum on a grandfather's clock is approximately 5 ft long.

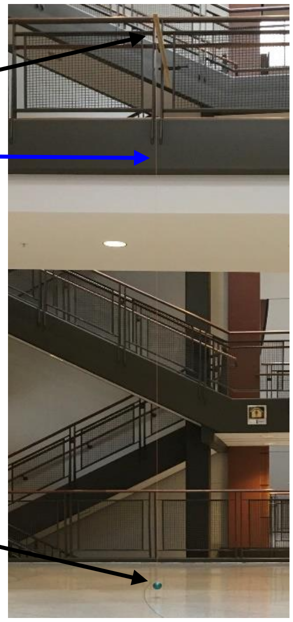
Use of stopwatch video app to determine times is provided in resources of this module at Hardware Store Science.org

Classroom Demonstration

You can try measuring the velocity of a pendulum yourself by setting up a tall support, attach a weight to a string and then attach the string to your support. Place a piece of cardboard with marks every inch behind the mass. Cause the pendulum to oscillate and record the motion of the pendulum. The recording can be slowed down so see the time the mass passes each distance on the cardboard. You now have the pieces to calculate the velocity. Hint: make sure that the point of attachment on the support stand is rigid –wiggling of the pendulum while oscillating will cause accuracies.

Top of Pendulum

Pendulum Wire
(hard to see in image)

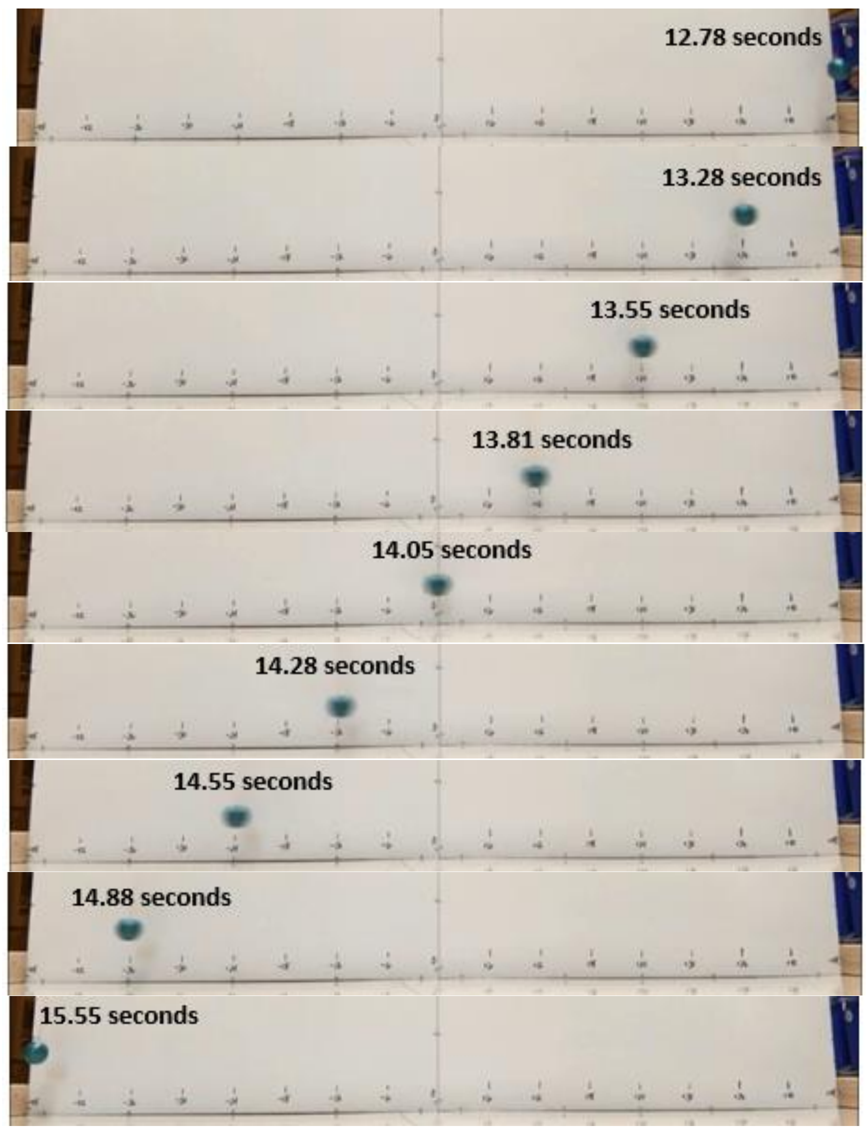


Pendulum bob
of 91 g mass

Measuring the velocity at various points during the pendulum's oscillation becomes easier as the period of oscillation increases. Data from a 32 ft long pendulum with a 0.091kg bob were measured in the atrium of Forney Hall at Purdue University. Images of the various components of the pendulum are shown to the right and below. The displacement as a function of time was recorded using a cell phone video and then the time was determined at each horizontal location B of pendulum's mass using the stopwatchcam app. A collage of the pendulum as it passed the fiduciary marks that were $\Delta B = 6$ inches apart is shown below along with a spread sheet showing a portion of the data. The distance B is zero when the pendulum is at rest, is +48 inches at the far-right of the pendulum's oscillation and is –48 inches at the far left of the oscillation.

Fiduciary - *(adjective)* used as a standard or reference of measurement

For the full video with time markers, see Module 4 resources on the Hardware Store Science.org website



Velocity and Potential Energy of Forney Atrium Pendulum							
Distance B_i, (in)	Time t_i, (sec)	Δt_i = $t_i - t_{i-1}$, (sec)	V_i = $\Delta B/\Delta t_i$, (m/sec)	B_{avg} = $(B_i+B_{i-1})/2$, (m)	h_i, (m)	PE_i, (J)	v_i^2, $(m/s)^2$
+48	12.78	0	0	0	0.076	0.068	0
+36	13.25	0.47	0.64	1.07	0.043	0.038	0.41
+24	13.55	0.3	1.02	0.76	0.019	0.017	1.04
+12	13.81	0.26	1.17	0.46	0.005	0.004	1.37
0	14.05	0.24	1.27	0.15	0.000	0.000	1.61
-12	14.28	0.23	1.32	-0.15	0.005	0.004	1.74
-24	14.55	0.22	1.38	-0.46	0.019	0.017	1.9
-36	14.88	0.25	1.22	-0.76	0.043	0.038	1.49
-48	15.55	0.67	0.45	-1.07	0.076	0.068	0.2
-36	16.27	0.72	0.42	-1.07	0.043	0.038	0.18
-24	16.61	0.34	0.89	-0.76	0.019	0.017	0.79
-12	16.88	0.27	1.13	-0.46	0.005	0.004	1.28
0	17.14	0.26	1.17	-0.15	0.000	0.000	1.37
+12	17.41	0.26	1.17	0.15	0.005	0.004	1.37
+24	17.71	0.3	1.02	0.46	0.019	0.017	1.04
+36	18.08	0.37	0.82	0.76	0.043	0.038	0.67
+45	18.71	0.63	0.48	1.03	0.066	0.059	0.23

In the spreadsheet above, the time difference Δt_i, average horizontal displacement $B_{avg,i}$ of the pendulum and the speed v_i between each of the fiduciary marks are determine as:

$$\Delta t_i = t_i - t_{i-1}$$

$$B_{avg,i} = \frac{B_i + B_{i-1}}{2}$$

$$v_i = \frac{\Delta B}{\Delta t_i}$$

where B_i is the horizontal displacement of the pendulum and the subscript refers to the i-th data point.

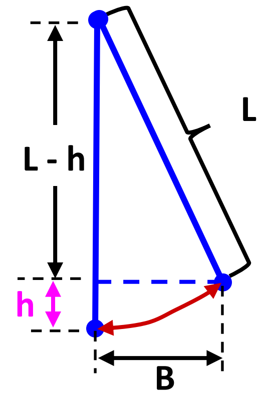
The next step is to determine how high the pendulum's mass has been raised for a given B_i, because this will allow us to determine the change in potential energy during the oscillation. A sketch of the relevant geometry of a pendulum of length L shows that when the horizontal displacement from the equilibrium position is B, then the vertical displacement is h. By the Pythagorean theorem

The difference between the vector velocity v and the magnitude of the velocity vector v was discussed in the Speed Velocity Acceleration Module.

$B^2 + (L-h)^2 = L^2$

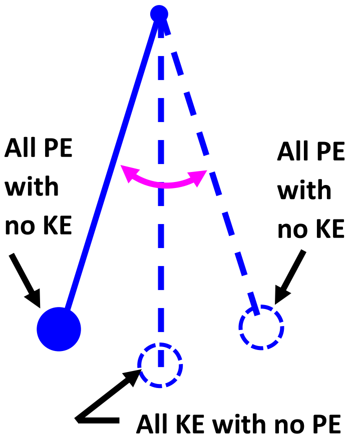
which upon rearrangement becomes

$$B^2 = 2Lh\left(1 - \frac{h}{L}\right) \approx 2Lh$$



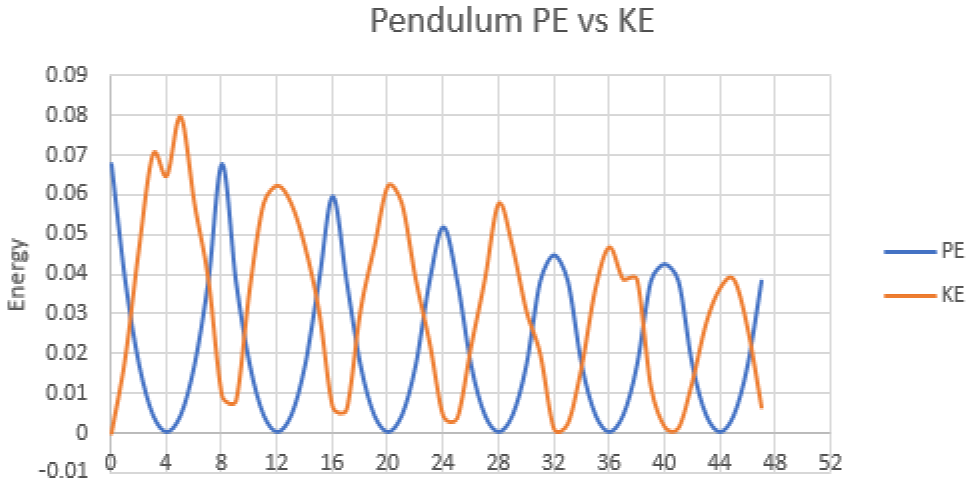
Since h/L is much less than one for small oscillations of the pendulum, $1 - h/L$ is approximately 1. And solving for h

$$h \approx \frac{B^2}{2L}$$



By measuring B and knowing L, the value of h at every B can be calculated. The values of h_i for each B_i are given in the Table above. The change in gravitational potential energy is then determined from $PE_i = mgh_i$, where m is the mass of the pendulum bob. It is important to understand that the potential energy is relative to the equilibrium position of the pendulum when it is at rest, where the gravitational potential energy is defined to be zero.

The potential energy and the velocity are plotted as a function of the horizontal displacement B for two oscillations. The potential energy undergoes an oscillation that is a maximum at the leftmost and rightmost excursions of the pendulum and is zero at the equilibrium position when $B = 0$. The velocity also undergoes an oscillation; however, it is a maximum when the pendulum is at the equilibrium position of $B = 0$ and is zero at both the leftmost and rightmost excursion of the pendulum. When the potential energy of the pendulum bob is at its maximum, the velocity is zero. When the potential energy is zero at the equilibrium position, the velocity is at its maximum. Thus, when the pendulum bob maximum. Thus, when the pendulum bob is at the right-most position all of the energy is in the form of potential energy, because the velocity is zero; and, when the potential energy is zero at the equilibrium position all the energy is in the form of kinetic energy. The pendulum is fully exchanging kinetic energy with potential energy every time the pendulum oscillates from it maximum excursion on the left or right to its equilibrium position.



The data in the figure above shows that during pendulum motion the potential energy is converted into kinetic energy. This leads to the obvious question, what is the functional form of kinetic energy. **Dimensional analysis** will provide the answer The dimensional units associated with gravitational potential energy are given by

Dimensional Analysis was discussed during lesson 3 of the Speed Velocity Acceleration Module

$$PE_{gravity} = mgh[=]kg \times \frac{m}{s^2} \times m = \frac{kg \times m^2}{s^2}$$

The symbol [=] is used in dimensional analysis to indicate 'equality' of units, where the actual values are disregarded, for example a mass m of 31.3 Kg, $m[=]kg$, neglecting the 31.3 value. Since potential energy, PE, and kinetic energy, KE, are added together as components of the total energy, they must have the same units; specifically, $PE[=]KE$. The kinetic energy must include at least the speed v and the mass m; however, we do not know the functional form, e.g. is it v^2 or v^3 or etc. Only thee integer powers of v or m are allowed, because a fractional power of a unit makes no sense, e.g.$meter^2$ is well defined vs. $\sqrt{meter}$ that has no meaning. Since the functional form of v or m in the kinetic energy are unknown, we can assume the general relationship, where the dimensions of $PE[=]KE$; thus

$$PE_{gravity}[=]\frac{kg \times m^2}{s^2}[=]m^a v^b[=]kg^a(m/s)^b$$

since $m[=]kg$ and $v[=]m/s$. The solution of requires $a = 1$ and $b = 2$. Thus,

$$KE = \alpha m v^2[=]mv^2$$

where α is a scalar constant. This is a pretty amazing result! From just dimensional considerations – the kinetic energy goes as the first power of the mass m and as the square of the speed v.

Finally, we need to determine the coefficient α. The pendulum data provides the needed information. Since the kinetic energy is zero at the maximum displacement B_{max} and the total energy in the system is constant at all values of the horizontal displacement B

$$E_{total} = \overbrace{\underbrace{PE}_{mgh_{max}} + \underbrace{KE}_{0}}^{\textbf{energy at } B_{max}} = PE_{max} = \underbrace{PE(B) + KE(B)}_{\textbf{energy at arbitra}}$$

Upon rearrangement and using the equation from earlier that relates the height h of the bob to its displacement B at any point during the pendulum's oscillation

$$\underbrace{PE_{max} - PE(B) = mg(h_{max} - h) = \frac{mg}{2L}(B_{max}^2 - B^2)}_{\textbf{potential energy}} = \underbrace{\alpha m v^2}_{\textbf{kinetic energy}}$$

A plot of $mg(B_{max}^2 - B^2)/2L$ versus the mass m times the square of the speed (v^2) at that B is shown to give an α essentially 1/2.

$$KE = 0.5mv^2$$

In most physical science courses, the expression for kinetic energy given above is just pulled out of thin air and we are told to memorize the formula and then use it to solve problems. This is because it is quicker and easier than the discovery process. The development in this Lesson, while time consuming, introduced (i) how a pendulum operates, which is one of the key mechanical devices in physical science, and (ii) dimensional analysis, where enforcing dimensional consistency is a key principal. These two principles have allowed us to discover the expression for kinetic energy rather than just read it and memorize a formula.

Lesson 2: Objectives

- **Define projectile motion based on horizontal and vertical velocity and displacement.**

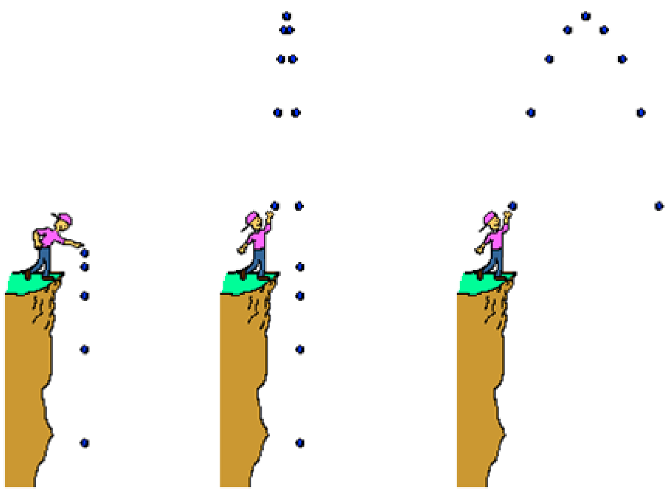
Content

- **Projectile Motion**
- **Interconversion of kinetic and gravitational potential energy**
- **Application of the equations of projectile motion**
- **The tangent function and uses**

Lesson 2: Characteristics of Projectile Motion

Once a projectile is set in motion it continues in that motion by its own inertia and is influenced by the downward force of gravity. It doesn't matter whether that object has been dropped, thrown straight up, or thrown at an angle. In the simplest analysis a projectile has only a single force that acts upon it - the force of gravity. Real projectiles almost always are influenced by friction with the surrounding fluid that slows down the motion (except if the projectile is in outer space which is a nearly perfect vacuum). Thus, a **free-body diagram** of a projectile (neglecting the effects of friction) would show a single force acting downwards as the force of gravity $F_{gravity}$. This downward force of gravity would be the same regardless of whether a projectile is moving upwards, downwards, or at an angle to the horizon.

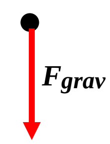
A free-body diagram is a graphical illustration of all of the forces on a body.



Object dropped

Object thrown straight up

Object thrown at an angle



Free body diagram of a projectile without friction. Note the free body diagram has the force pointing downward, since gravity is the only force acting on the projectile.

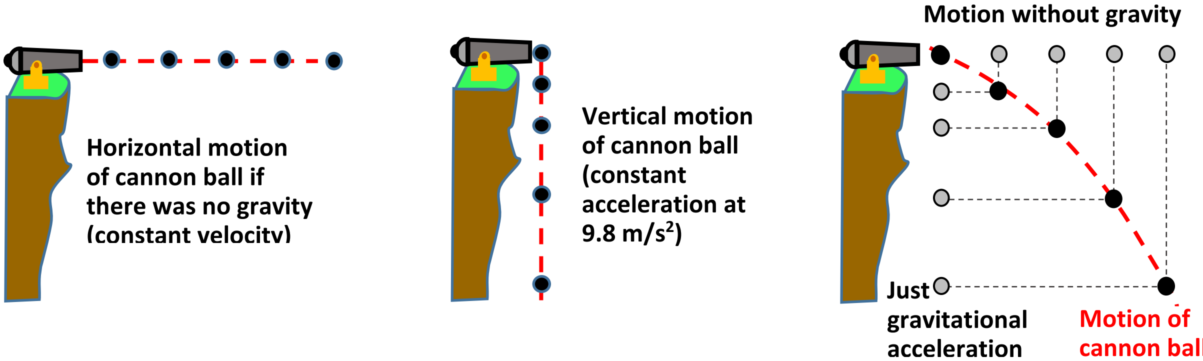
The fact that projectiles undergo motion in the horizontal x-direction and in the vertical y-direction make them unique in many respects. When there are no frictional forces on the projectile, these two perpendicular components of motion are independent of each other and can be evaluated separately. To help us with this process lets review the horizontal and vertical components of an object's motion, focusing on the presence/absence of forces, acceleration and velocity.

For a more in-depth analysis of forces acting perpendicular to one another refer to the Pulley Module

Horizontally Launched Projectiles

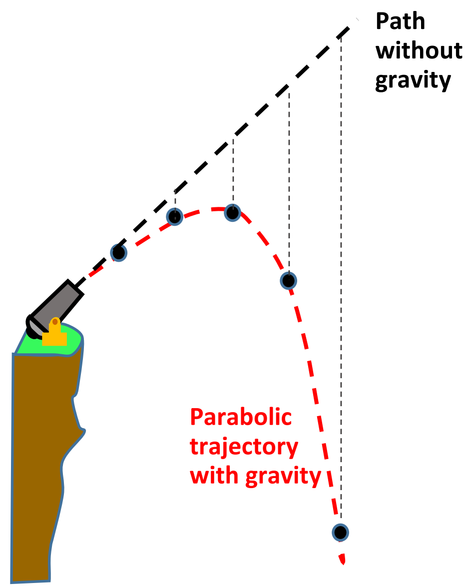
Consider a cannonball shot horizontally by a cannon from the top of a very high cliff. In the absence of gravity, the cannonball would continue its horizontal motion at a constant velocity in accordance to Newton's 1st law of inertia. If, on the other hand, the cannonball was just dropped from the cliff in the presence of gravity, the cannonball would accelerate downward, gaining speed at a rate of $9.8m/s$ every second, this is just the **acceleration due to gravity** on Earth with $g = 9.8\,m/s^2$.

If the cannonball is fired horizontally in the presence of gravity, the cannonball will travel horizontally at a constant velocity while simultaneously accelerating downward at a rate of $9.8m/s^2$. The presence of gravity does not affect the horizontal motion of the cannonball, because the force of gravity only acts downward and not in opposition to the horizontal motion of the cannon ball. This explains the perpendicular components of motion being independent of each other.



Non-Horizontally Launched Projectiles

Now suppose that the cannon is aimed at an upward angle and shot from the same cliff. In the *absence of gravity* there are no external forces on the projectile and thus inertia requires that the projectile would travel in the same upward angle upon which it was fired. The cannonball would travel upwards with both a horizontal and vertical component of its velocity as shown by the dashed line in the diagram to the right.



In the presence of gravity, however, the projectile would travel with a *parabolic trajectory*, a specific curve shaped like an arch. Initially the cannon ball would rise, but the force of gravity causes the cannon ball to accelerate downward. Since the cannonball was shot from the edge of a cliff, gravity will eventually cause the cannonball to drop even lower than the position from which it was shot. Since the horizontal and vertical motion are independent, the effect of gravity's downward force acting upon the cannonball causes the same vertical motion as if it were simply dropped. This vertical force of gravity acts on the cannon balls vertical motion; however, gravity has no effect on the cannonballs horizontal motion.

To summarize, we have learned the following about projectiles.

- The horizontal velocity of a projectile is constant, never changing in value.
- The horizontal motion of a projectile is independent of its vertical motion.
- There is a downward vertical acceleration of $-9.8m/s^2$ caused by gravity. The minus sign is because gravity is in the downward direction.
- Projectiles travel with a parabolic trajectory due to the influence of gravity.

Interconversion of Potential and Kinetic Energy in Projectiles

Refer to the Energy Storage Module for a discussion of the Conservation of Energy principle and the various forms of energy.

The description above of projectile motion was in terms of how gravitational force affects the vertical, but not the horizontal, motion of the projectile. As soon as a projectile is released, as in the cannonball leaving the mouth of the cannon, we consider the cannonball as an isolated system. An **isolated system** is one where no work or heat passes through the boundary between the system and surroundings. The **Conservation of Energy** principle for an isolated system states that the total energy in the system is constant. This is due to the fact that total energy is the sum of the potential energy, kinetic energy and thermal energy acting on the system.

Total Energy = Potential Energy + Kinetic Energy + Thermal Energy = Constant

Since the temperature of the projectile is constant, the thermal energy is constant; thus, the sum of the potential energy and kinetic energy must remain constant as well. We define the potential energy as composed of the following components:

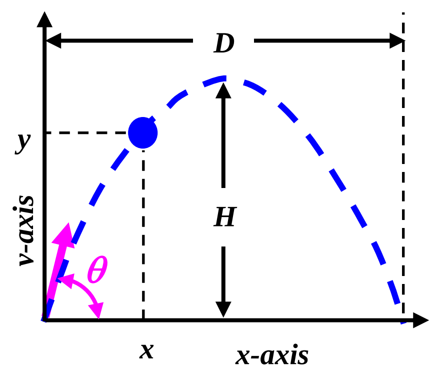
Potential Energy = Gravitational Energy + Stored Mechanical Energy + Stored Chemical Energy + Stored Electrical Energy

In a projectile there is no stored mechanical, chemical or electrical energy, only the gravitational potential energy. Thus, for a projectile (which in the absence of friction is an isolated system)

Total Energy = Gravitational Potential Energy + Kinetic Energy = Constant

Remember that the velocity vector $\boldsymbol{v_o}$ is indicated by the bold typeset. v_o, which is not bold, is the speed, where the speed is the magnitude of the velocity vector $\boldsymbol{v_o}$

Consider a projectile of mass m that is launched from the point $(x = 0,\ y = 0)$ with an initial velocity $\boldsymbol{v_o}$ as shown in the diagram, where θ is the angle at which the projectile is launched. The horizontal displacement of the projectile is x and the vertical displacement is y. Because the total energy of projectile is constant, the energy at launch equals the energy during the flight of the projectile. Thus, the total energy of the projectile is given by



$$E_{total} = constant = \overbrace{\underbrace{PE_o}_{0} + \underbrace{KE_o}_{mv_o^2/2}}^{\text{at launch}} = \overbrace{\underbrace{PE}_{mgy} + \underbrace{KE}_{mv^2/2}}^{\text{during flight}}$$

where the initial kinetic energy is $mv_o^2/2$, the kinetic energy during the flight of the projectile is $mv^2/2$, and v is the velocity of projectile during flight. The potential energy during flight is mgy. The projectile velocity can be written in its two components v_x and

v_y , where the two components of the initial velocity $\boldsymbol{v_o}$ are v_{ox} and v_{oy} and the velocity during the flight is v_x and v_y. If we remember that the horizontal velocity is constant for a projectile when there is no friction then the x components are $v_x = v_{ox} = constant$. This simplifies defining relationship for Conservation of Energy for a projectile that experiences no friction. We can derive the y component of velocity for the cannonball by rearranging the resulting equations as

$$\frac{1}{2}v_{oy}^2 = \frac{1}{2}v_y^2 + gy$$

Then,

$$v_y = \sqrt{v_{oy}^2 - 2gy}$$

Remember $y = 0$ when the projectile is initially launched. When the projectile is just launched the cannonball begins moving upwards and the small vertical change during the initial part of the trajectory results in the quantity under the square root (√) will be a little smaller than the initial velocity (v_{oy}^2), causing the trajectory to curve downward as shown in the schematic.

Let see if we can simplify these observations a little. The key ideas expressed in equations above are:

1. The only energy in the projectile is its kinetic and gravitational potential energy,
2. The total energy is the sum of the kinetic energy and gravitational potential energy is constant throughout the projectiles motion.
3. When the projectile is launched it only has kinetic energy, because the height h is defined to be zero at the launch point.

$$PE = 0$$

$$KE = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2$$

4. During flight, some of the projectile's kinetic energy at launch is converted to gravitational potential energy.

$$PE = gy = \Delta\left(\frac{1}{2}mv_y^2\right)$$

$$KE = \frac{1}{2}mv_x^2 + \Delta\left(\frac{1}{2}mv_y^2\right)$$

These four points are the most important features of projectile motion. Everything else is just turning these four points into a mathematical calculation. Some secondary points are:

5. The velocity in the horizontal direction (v_x) remains the same throughout the flight.
6. The vertical component of the velocity v_y changes during flight of the projectile, v_y decreases as the projectile goes up and increases as the projectile comes down.
7. At the highest point in the projectile's trajectory the vertical component of the velocity is zero, i.e. $v_y = 0$. The maximum projectile height is $H = v_{oy}^2/2g$

The equations in the previous paragraph are just ways to precisely state the points above (1-7). Ideas in words and ideas in equations go hand-in-hand – scientists and engineers view words without equations as being without precise meaning and equations by

Remember that the velocity vector $\boldsymbol{v_o}$ is indicated by the bold typeset. v_o, which is not bold, is the speed, where the speed is the magnitude of the velocity vector $\boldsymbol{v_o}$

In projectile motion some of the kinetic energy of the projectile at launch is converted into gravitational potential energy; however, the

total energy remains constant.

For a more in-depth analysis of forces acting perpendicular to one another refer to the Pulley Module

themselves without interpretation as just math not science or engineering – we need both words and equation to fully understand motion.

The determination of the trajectory of a projectile has been complex, because we need to determine how v_y depends upon the height of the projectile at any given time – this requires the solution of a differential equation which in turn requires calculus. While we will not go that far we should become familiar with this analysis as we explore further. The resulting equations defining projectile motion with no frictional effects are summarized below:

Equations of Projectile Motion

Trajectory: $y(t) = v_{oy}t - \frac{g}{2}t^2$ **and** $x(t) = v_{ox}t$ **OR** $y = \frac{v_{oy}}{v_{ox}} \times \left(-\frac{1}{2}g\frac{1}{v_{ox}^2}x^2\right)$

Projectile Maximum: $H = \frac{v_{ox}^2}{2g} = \frac{gt_H^2}{2}$; $t_H = \frac{v_{oy}}{g}$; $x_H = \frac{v_{ox}v_{oy}}{g} = v_{ox}\sqrt{\frac{2H}{g}}$

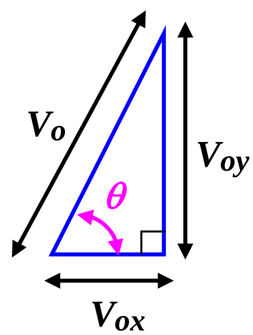
Maximum Distance: $D = \frac{2v_{ox}v_{oy}}{g} = t_D v_{ox}$; $t_D = \frac{2v_{oy}}{g}$

The tangent function is quite useful in computing aspects of projectile motion.

Using your smartphone, do the following steps:

1. **Open science calculator**
2. **Click 2nd button**
3. **Enter 10 ÷ 5 = tan^{-1}**
4. **Degree value will be displayed**

These equations are extremely helpful when analyzing projectile motion. Measuring the maximum height H and the time t_H it takes the projectile to reach this height allows us to calculate the y-component of the initial velocity v_{oy}. By measuring the distance D and the time t_D it takes the projectile to return to its starting height of $y = 0$, we can determine the x-component of the initial velocity v_{ox}. The launch angle θ can be determined from the relationship of the sides of a right triangle



$$\tan\theta = \frac{v_{oy}}{v_{ox}}$$

The tangent function $\tan\theta$ is a tabulated function that allows us to determine the launch angle. For example if $v_{oy} = 10m/s$ and $v_{ox} = 5m/s$, then we can rearrange this equation as

$$\theta = \tan^{-1}\frac{v_{oy}}{v_{ox}} = \tan^{-1}\frac{10\,m/s}{5\,m/s} = 63.4^o$$

You could also look up the $\tan\theta$ value in a table.

Here is another use of the tangent function:

Sample Problem

If a projectile is launched with a total velocity (i.e. speed) of $15m/s$ at a 60^o angle, what are v_{ox} and v_{oy}?

We can rearrange the $\tan\theta$ equation to isolate v_y (or v_x)

$$v_y = \tan\theta\, v_x = \tan 60 v_x = 1.732v_x$$

We then use the Pythagorean theorem to solve for v_x

$$v^2 = v_x^2 + v_y^2 = v_x^2 + 1.732v_x^2 = (1 + 1.732^2)v_x^2$$

Re then rearrange solving for v_x,

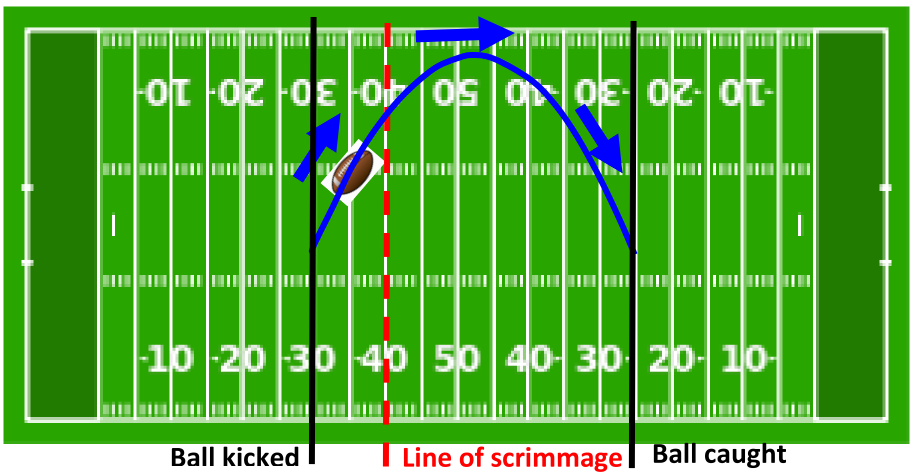
$$v_x = \sqrt{\frac{v^2}{1 + 1.732^2}} = \sqrt{\frac{(15\,m/s)^2}{1 + 1.732^2}} = \sqrt{\frac{225\,m^2/s^2}{3.999}} = 7.5\,m/s$$

We can then go back and solve for v_y

$$v_y = 1.732 v_x = 1.732 \times 7.5\,m/s = 13\,m/s$$

Example of Projectile Motion

Question: The punter on your school's football team stands 10 yards behind the line of scrimmage and is able to punt the ball 35 yards in front of the line of scrimmage, where the football has a hang time (i.e. how long it is in the air) of 4 seconds. What is the angle and initial velocity of the ball (and hence the punter's foot)? You can assume that there no frictional losses with the air during the flight of the ball.



Answer: The time t_D that the ball stay in the air is 4s; thus,

$$v_{oy} = \frac{t_D g}{2} = \frac{4s \times (32ft/s^2)}{2} = 64ft/s^2$$

The distance D that the ball travels is $10 + 35 = 45 yards = 135ft$, and by

$$v_{ox} = \frac{Dg}{2v_{oy}} = \frac{135ft \times 32ft/s^2}{2 \times 64ft/s} = 33.75ft/s$$

and thus

$$v_o = \sqrt{v_{ox}^2 + v_{oy}^2} = \sqrt{33.75^2 + 64^2}\,ft/s = 72.3\,ft/s$$

Finally, the angle is determined as

$$\theta = \tan^{-1}\frac{v_{oy}}{v_{ox}} = \tan^{-1}\frac{64\,ft/s}{33.7\,ft/s} = 52.2^o$$

Question: The coach would like to increase the hang time of the punt to 5 seconds so that his team has sufficient time to reach the punt returner before he takes off running with the ball. How much does the initial velocity of the ball is need to increase and does there need to be a new launch angle?

Answer: The analysis proceeds like above but now with

$$v_{oy} = \frac{t_D g}{2} = \frac{5s \times (32ft/s^2)}{2} = 80ft/s^2$$

The distance D still is 135 ft so,

$$v_{ox} = \frac{Dg}{2v_{oy}} = \frac{135ft \times 32ft/s^2}{2 \times 80ft/s} = 27ft/s$$

and thus

$$v_o = \sqrt{v_{ox}^2 + v_{oy}^2} = \sqrt{27^2 + 80^2}\, ft/s = 84.4\, ft/s$$

Which requires a $(84.4 - 72.3)/72.3 = 17\%$ increase in the initial velocity. Finally, the new angle is

$$\theta = \tan^{-1}\frac{v_{oy}}{v_{ox}} = \tan^{-1}\frac{80\, ft/s}{27\, ft/s} = 71.35^o$$

Lesson 3: Objectives

- **Investigate projectile motion and relate data collected on rubber band stretch to parabolic trajectory of projectile.**

Content

- **Measure trajectory of a projectile**
- **Calculate the PE and KE of a projectile**
- **Determine interconversion of PE and KE**
- **Determine PE stored in rubber band**

Lesson 3: Projectile Motion with a Ping-Pong Ball Launcher

An excellent way to measure projectile motion is with a ping-pong ball launcher powered by a rubber band. The ball launcher should allow for the launching of a ball at different angles and with different amounts of stored energy in the rubber band.

Construction of Ball Launcher

One way in which to build a ping-pong ball launcher would be to create a base and frame that controls the angle of a flat board. We could add two eye screws to attach the rubber bands. If we add lines to the board so that we know how far the rubber bands have been pulled back prior to launching projectile we can determine force. Attaching the launch platform to the launcher stand by a pair of thumb screws allows us to change the angle of the board by first loosening and then re-tightening the thumb screws. We can attach one end of the two rubber bands to eye screws and the other end of the rubber bands can be held together by a short piece of string. The string will allow us to pull back the rubber bands and then release the ball in a controlled manner. A small cradle into which the ball is placed could also be made from a small cup which has two holes through which the rubber bands may pass prior to connecting them with a short piece of string.

A more complete description of the building process for the ping-pong ball launcher is available on the Hardware Store Science.org website.

Steps for construction of the ping-pong ball launcher include:

1. Cut a square piece of ¾ plywood for a base, 2 pieces of 2 x 4 for upright supports, and on piece of 1 x 4 for the launching platform.
2. Mount the 2 x 4 supports to the plywood base using screws, so that the 1 x 4 platform fits between them and is centered on one edge of the base. Screws should

be driven in from underside of base, through one end of support. Predrilling holes will make driving screws easier.

3. Securely attach 2, 1-1/2 x 5/8-inch corner braces to one end of 1 x 4 platform with screws. Predrilling holes will help here as well.
4. Drill a hole through the top end of support. Center the hole 1 inch from top and attach launch platform using 1/4-20 bolts through corner brace and hole in support. Secure bolt with 1-inch flat washer and wing nut.
5. Attach 2 eye bolts to the 1 x 4 launch platform end secured to the upright supports. Attach a rubber band to the eye bolts and secure a “pull String” to the center of the rubber band.

A video of the construction process is found in Module 7 Resources on the Hardware Store Science.org website

Initial Investigation of Projectile Motion

When we initially begin investigating a phenomenon, it is useful to determine how the various variables under control by the experiment affect the basic features that are observed. In the ball launcher experiment the control variables are:

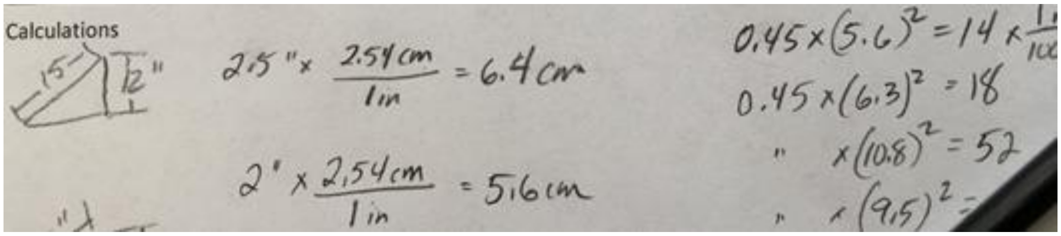
- The angle of the launcher
- How far the rubber bands are pulled back in order to launch the ball
- The type of rubber band, i.e. two thick rubber bands vs. two thin rubber bands
- The diameter of the ball
- The surface of the ball e.g. is it a smooth ping-pong ball or does it have holes in the ball like a whiffle ball.

There is often an art to an experiment, where with some practice you can produce better, more reproducible data.

The simplest thing to measure is how far the ball goes before it hits the floor. The example lab notebook below shows actual distances traveled for various angles and stretch lengths of the rubber band. There is an art to releasing the ball, where the string on the ball launcher assembly needs (i) to be parallel to launching board and (ii) cleanly let go - this is the only way to obtain reproducible data. Also, if the ball has a higher trajectory it is often more accurate (this is why good basketball shots often have a high arc).

Data Table 1

Trial	Launcher Angle	Rubber Band Stretch (l)	Force ($F = k \times l^2$)	Distance to “target”
1	53°	6.4 cm	0.18 N	20 feet
2	42°	5.6 cm	0.14 N	20 ft
3	30°	6.3 cm	0.18 N	20 ft
4	53°	10.8 cm	0.52 N	15 ft
5	42°	9.5 cm	0.41 N	15 ft
6	30°	10.2 cm	0.47 N	15 ft
7	53°	5.1 cm	0.12 N	25 ft
8	42°	3.8" cm	0.06	25 ft
9	30°	5.1 cm	0.12 N	25 ft



Inquiry Experiments

Once you have completed your first experiment, you will have all of the tools necessary to answer additional projectile motion questions. Questions that would be interesting to explore include

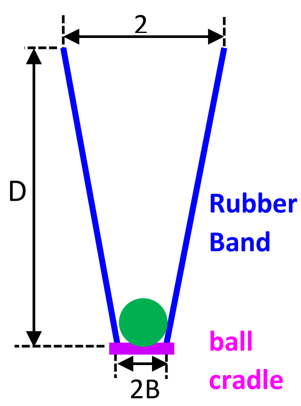
1. What happens if the target is moved up and down in the vertical direction instead of forwards and backwards in the horizontal direction? Compare the results of vertical target displacement with horizontal target displacement.
2. Using targets of varying size, how does the size of the target relate to the launch variables of angle and how far the rubber band is pulled back? Note: it is OK if an investigated variable has little or no effect.
3. Will the results be the same for two cases where the total cross-sectional area of rubber bands is the same but the number of bands is different? (Note: a cut rubber band tied at the ends to the frame will have half the cross sectional area of an uncut rubber band.)
4. Find rubber bands of the same cross-sectional area but of different lengths. Does the starting length of the rubber band have an influence on the displacement or amount of energy stored?

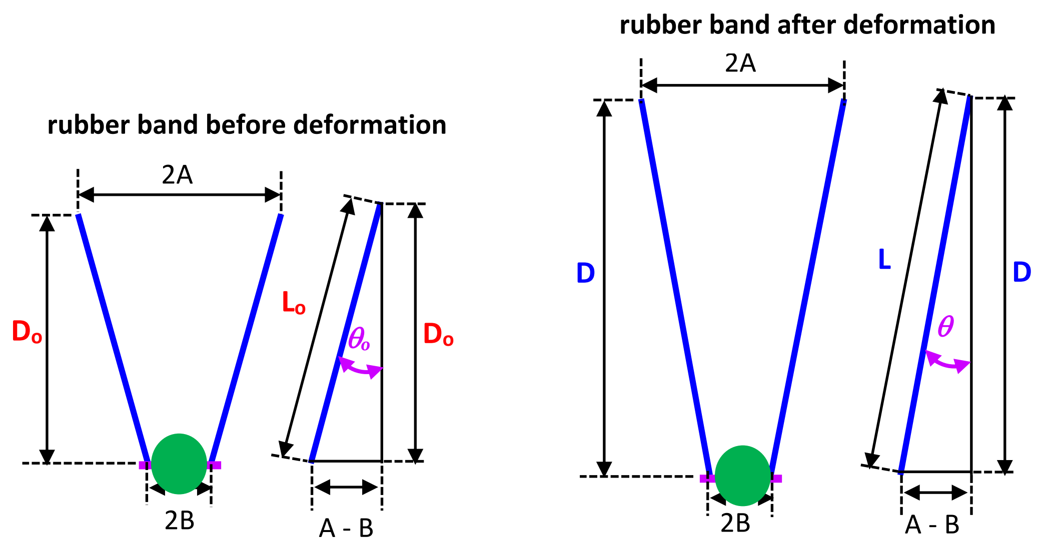
Fun Activity

After you have had time to collect **and organize** data from your ball launcher, you might challenge others to a contest to see if they can hit a bucket at a known distance from the launcher. The challenge could include the maximum distance the bucket can be from the launcher. Based upon previous data, each team will adjust the angle and how far the rubber bands are pulled back, and be given three attempts to put the ball into the bucket.

Potential Energy Stored in the Rubber Bands

After playing with the ball launcher we now knows important information about the trajectory of the ball, but what about the energy stored in the rubber bands? We have already learned about energy stored in a rubber band launcher in the Energy Storage module, during the car launcher experiment. The ping-pong ball is essentially the same as the rubber band car launcher experiment. Let's repeat the analysis here for the rubber bands we used in this experiment. However, if you used the same rubber bands in the ping-pong ball launcher and the car launcher experiments you have already completed this analysis.





The schematic above defines the relevant distances that will need to be determined. First, place the ball on launching platform and determine the distance D_o when the rubber band is tight, but not stretched, i.e. no wrinkles in the rubber band – the length of the rubber band should now be L_o based on the Pythagorean Theorem

$$\boldsymbol{L_o^2} = (A - B)^2 + \boldsymbol{D_o^2}$$

$$\boldsymbol{L_o} = \sqrt{(A - B)^2 + \boldsymbol{D_o^2}}$$

The distance A and B should be measured and recorded. With the information of D_o we can calculate L_o – we can check our calculations with a measurement of L_o. Now consider the situation when the launcher is moved from its unstretched location D_o to the stretched location D. Again, apply the Pythagorean Theorem

$$\boldsymbol{L^2} = (A - B)^2 + \boldsymbol{D^2}$$

$$\boldsymbol{L} = \sqrt{(A - B)^2 + \boldsymbol{D^2}}$$

Thus, the change in length of the rubber band as a function of D, i.e. $\Delta L(D)$ is

$$\Delta L(D) = \boldsymbol{L} - \boldsymbol{L_o} = \sqrt{(A - B)^2 + \boldsymbol{D^2}} = \sqrt{(A - B)^2 + \boldsymbol{D_o^2}}$$

This calculation involves a little algebra, but it is straightforward.

The next thing that we need to calculate is the potential energy stored in the rubber bands as a function of the distance D that the launcher has been pulled back. For a material that obeys Hooke's Law the potential energy stored in a rubber band is $PE = k\Delta l^2/2$. While this is the result for one rubber band, for the two rubber band launcher system the total potential would be $k\Delta l^2$, where rearrangement of this equation allows us to calculate Δl as a function of the distance D. Thus, we now have the potential energy stored in the rubber band powered launcher system just prior to launching the ping-pong ball.

What now needs to be determined is how much of this stored energy is converted into potential energy as the ping-pong ball exits the platform. Set the platform at a given angle, launch the ball and determine how high it travels for a given displacement D of the launcher. Record your results in your lab notebook. From the displacement determine the change in height Δh that the ball undergoes. Repeat the experiment three times and take an average of the distance from the platform as well as the average height of the ball. Remember the formula for the change in potential energy due to an objective changing its height under the influence of gravity, i.e. $\Delta PE = mg\Delta h$, where m is the mass of the object. Measure the mass of the ball and then compute the ΔPE_{ball}; then determine the for the $PE_{rubber\ band}$ displacement D that was used to launch the car. What is the fraction of the PE stored in the rubber band that was recovered in the change in PE resulting from the height change of the ball as it left the platform? Repeat the experiment for three different launcher displacements D and two different ramp angles. Record all these calculations either in your lab notebook OR if you make the calculations in a spreadsheet, then paste a printout of the spreadsheet in your lab notebook.

You can compute PE_{stored} for each of your ping-pong ball launches and add this as a final column in the spreadsheet of your data.

Lesson 4: Objectives

- **Explain how time of flight influences the horizontal and vertical displacement of a projectile.**

Content

- **Determine the motion of projectile from features of its trajectory**

$x(t)$ – represents the horizontal displacement as a function of time

$x(t)$ – represents the vertical height as a function of time

Downstream – (*adjective*) of or pertaining to the latter part of a process or system

Lesson 4: Projectile Motion Analysis

The characteristics of projectile motion were developed in Lesson 2 and in Lesson 3 projectile motion was experimentally investigated using the rubber band powered ping-pong ball launcher. We will now apply this new understanding to describe various features of projectile motion. No new information will be presented in this lesson, rather we will view this as is an opportunity to apply the concepts learned during the previous lessons.

When we launch a projectile at any angle θ with an initial velocity $\boldsymbol{v_o}$ as shown in the figure, we can use the symbol $x(t)$ to represent the horizontal distance of the projectile and $y(t)$ to represent the vertical distance, where both quantities change with time. We will also assume that the launch point of the projectile is the point (x, y) = $(0,0)$. Remember that velocity $\boldsymbol{v_o}$ is a vector, where it has a component in the x-direction v_{ox} and a component in the y-direction v_{oy}. The trajectory of the projectile (i.e. the dashed line that the projectile traces out in space as it moves) is defined by $y(t)$. The peak height of the projectile is H, which is reached at a time t_H after launch at a horizontal distance x_H downstream from the launch point. The projectile returns to the height at which it was launched at a horizontal distance D downstream from the launch point at a time t_D. It is important to point out that the projectile in some instances can go lower than its launch point, e.g. a cannonball that is fired from the top of a cliff as illustrated in Lesson 2.

Determination of the Time of Flight

We must always remember that the horizontal motion of a projectile is independent of its vertical motion, at least in the case where there is no friction from the surrounding fluid. The time for a projectile to rise vertically to its peak (as well as the time to fall from that peak) depends upon the vertical speed of the projectile. Determination of the time required for a projectile to reach its peak height H just requires the straightforward use of acceleration. Acceleration is the rate at which the velocity of an object changes in a given interval of time. The vertical acceleration due to gravity on the surface of the earth is $-9.8\, m/s^2$ which means that the vertical velocity decreases (because of the minus sign) every second by $9.8\, m/s$. If the acceleration is constant, the relationship between velocity v and acceleration a is simply

$$v_f = v_i + at$$

Where v_i is the initial velocity and v_f is the final velocity after a time interval t.

Velocity and acceleration were discussed in the Speed Velocity Acceleration Module.

We can apply the relationship between velocity and acceleration to the situation of a projectile that is moving upwards with an initial velocity of $39.2\,m/s$. As it travels upwards Its velocity will decrease by $9.8\,m/s$ every second due to gravity; thus, the projectile's vertical velocity after 1 second would be

$$39.2\,m/s + (-9.8\,m/s^2 \times 1s) \ = 29.4\,m/s$$

And after 2 seconds it would be

$$39.2\,m/s + (-9.8\,m/s^2 \times 2s) \ = 19.6\,m/s$$

And after 3 seconds

$$39.2\,m/s + (-9.8\,m/s^2 \times 3s) \ = 9.8\,m/s$$

And after 4 seconds the velocity would be

$$39.2\,m/s + (-9.8\,m/s^2 \times 4s) \ = 0\,m/s$$

For a projectile with an initial vertical velocity of $39.2\,m/s$ it would take 4 seconds for the projectile to reach its highest point, i.e. the peak in the figure above, where its vertical velocity is $0\,m/s$. The velocity would be zero for just an instant. Since the acceleration due to gravity is a constant, velocity will continue to decrease until at the peak $v_f = 0$. We can therefore do some simplifying and rearranging as follows

$$0 = v_f = \underbrace{\cancel{v_i}}_{v_{oy}} + \underbrace{\cancel{a}}_{-g}\underbrace{\cancel{t}}_{t_H} = v_{oy} - g t_H$$

Which results in the relationship given for the time t_H to reach the peak.

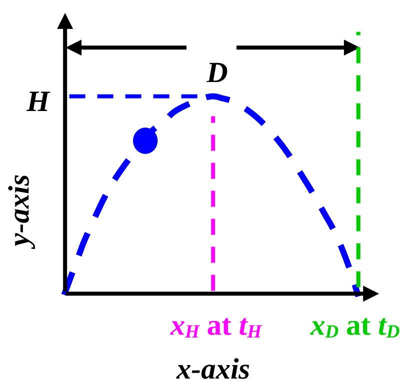
$$t_H = \frac{v_{oy}}{g}$$

And combining this relationship for t_H with the expression for $y(t)$ we get

$$H = y(t = t_H) = v_{oy} t_H - \frac{g}{2} t_H^2 = v_{oy}\left(\frac{v_{oy}}{g}\right) - \frac{g}{2}\left(\frac{v_{oy}}{g}\right)^2 = \frac{v_{oy}^2}{2g}$$

which gives the peak height H in terms of the y-component of the initial velocity.

The projectile only remains at its peak for an instant and then gravity causes the projectile to fall. Eqn. 4.24 still describes how the acceleration due to gravity effects the velocity. The velocity starts at 0 m/s at 4 seconds and then decreases by the gravitational constant *g*; thus, at 5 seconds the velocity is



$$39.2\,m/s + (-9.8\,m/s^2 \times 5s) \ = -9.8\,m/s$$

And at 6 seconds

$$39.2\,m/s + (-9.8\,m/s^2 \times 6s) \ = -19.6\,m/s$$

And at 7 seconds

$$39.2\,m/s + (-9.8\,m/s^2 \times 7s) \ = -29.4\,m/s$$

And finally, at 8 seconds

$39.2\,m/s + (-9.8\,m/s^2 \times 8s) \; = -39.2\,m/s$

Note that the peak H occurred at 4 seconds, and examining the trajectory it is symmetric around the peak; thus, the projectile returns to its initial height at $t_D \; = \; 2t_H$ at $8s$ with a velocity in the vertical direction that is the ***negative*** of its vertical launch velocity. The relationship between displacement time t_D and time to max height t_H is $t_D \; = \; 2t_H$. Also, by the symmetry of the projectile's trajectory $D = 2x_H$.

As an example, consider the case when a football is kicked, where it takes 2 seconds to rise to its peak. Then, it will take 2 seconds to fall from the peak, given a total flight time of 4 seconds.

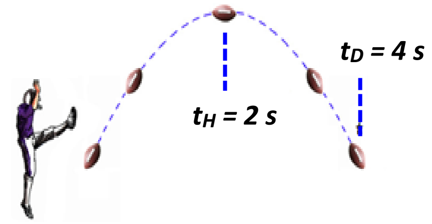
Determination of Horizontal Displacement

The horizontal velocity of a projectile in the absence of frictional effects is constant, where the horizontal displacement can be determined using the standard formula relating velocity v and displacement x

$v_f = v_i + at$

Where x_i is the initial displacement and x_f is the final displacement after time t.

Consider the football example again, where the projectile has a time of flight of 4 seconds. If the horizontal component of the velocity v_{ox} is 15 m/s, then the overall horizontal displacement would be $15ms \times 4s = 60m$.



Determination of Peak Height

The maximum height of the trajectory *H* is determined by

$$H = \frac{gt_H^2}{t} = \frac{9.8\,m/s^2 \times (2s)^2}{2} = 19.6m$$

where $t_H = 2s$

Determination of Velocity from H, D and $t_D = 2t_H$

By measuring key features of the trajectory of a projectile we can determine the components of the initial velocity, i.e. v_{ox} and v_{oy}.

$$v_{ox} = \frac{D}{t_D} = \frac{D}{2t_H} = D\sqrt{\frac{g}{8H}}$$

and

$$v_{oy} = gt_H = \frac{gt_D}{2} = \sqrt{2gH}$$

Thus, by measuring H, D and $t_D = t_H$ we can determine the initial velocity of the projectile.

Consider the football example again, where we have determined the following quantities: $t_H = 2s$, $t_D = 4s$, $H = 19.6m$ and $D = 60m$. First, we will determine the horizontal component of the initial velocity:

$$v_{ox} = \frac{D}{t_D} = \frac{60m}{4s} = 15\,m/s$$

or in an alternate calculation

$$v_{ox} = D\sqrt{\frac{g}{8H}} = 60m\sqrt{\frac{9.8\,m/s^2}{8 \times 19.6m}} = 60m\sqrt{\frac{1}{6}}m/s = 15\,m/s$$

And for the vertical component of the initial velocity

$$v_{oy} = gt_H = 9.8\,m/s^2 \times 2s = 19.6\,m/s$$

or in alternate calculation

$$v_{oy} = \sqrt{2gH} = \sqrt{2 \times 9.8\,m/s^2 \times 19.6m} = 19.6\,m/s$$

Finally, the magnitude of the velocity (i.e. the speed) is given by

$$v = \sqrt{v_{ox}^2 + v_{oy}^2} = \sqrt{(15\,m/s)^2 + (19.6\,m/2)^2} = \sqrt{(15^2 + 19.6^2)\,m^2/s^2} = \sqrt{225 + 384.2}\,m/s = 24.7\,m/s$$

Once the key features of the trajectory are known, the velocity can be calculated.

Note the units on the coefficients are important, so that when multiplied by x and x^2 the resulting units on *y* are consistently determined to be meters.

Note that when performing these calculations, the units (i.e. *m* and *s*) on all terms were included. This is important to make sure we have a consistent answer. For example. if instead of $g = 9.8\,m/s^2$ one used English units for $g = 32\,ft/s^2$ and metric for the distance *D = 15* m, then in the calculation for we would have had

$$\boldsymbol{v_{oy} = \sqrt{2gH} = \sqrt{2 \times 32\,ft/s^2 \times 19.6m} = \sqrt{1254\,ftm/s^2}}$$

The square root of a (*ft m*) makes no sense – we would need to convert feet to meters or meters to feet to have a reasonable answer. If however one drops the units from the calculations one would just have

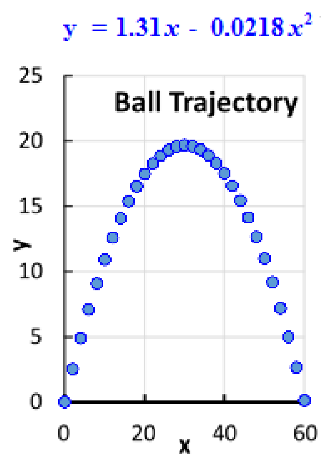
$$\boldsymbol{v_{oy} = \sqrt{1254} = 35.4}$$

which looks perfectly reasonable but makes no sense. Keeping the units is an important way to make sure you are using the correct quantities in a calculation.

Advanced Topic: Determination of the Complete Trajectory

The complete trajectory can be computed once the horizontal and vertical components of the velocity are known using Eqn. 4-17c. This is most easily done using a spread sheet. For the football example above, we are interested in the trajectory from x at $0m$, i.e. the projectile launch point, to $x = D = 60m$, i.e. the point the football hits the ground. As determined above $v_{ox} = 15\,m/s$ and $v_{oy} = 19.6\,m/s$. Thus, the coefficient of x in Eqn. 4-17c is $v_{oy}/v_{ox} = 1.31$, and the coefficient of x^2 is $-g/2v_{ox}^2 = -0.0218m^{-1}$ The calculation spreadsheet and associated graph are to the right

The trajectory is a parabola, which is defined by the equation $y = a + bx + cx^2$ (in this example $a = 0$, $b = 1.31$ and $c = -0.0218$ with the appropriate units).



x, m	y	x, m	y, m
0	0.0	32	19.6
2	2.5	34	19.3
4	4.9	36	18.9
6	7.1	38	18.3
8	9.1	40	17.5
10	10.9	42	16.6
12	12.6	44	15.4
14	14.1	46	14.1
16	15.4	48	12.7
18	16.5	50	11.0
20	17.5	52	9.2
22	18.3	54	7.2
24	18.9	56	5.0
26	19.3	58	2.6
28	19.6	60	0.1
30	19.7		

Summary

One of the important features of physical science is the ability to make predictions about the final outcome of a process – in this case the motion of a projectile. These predictions are made through the application of physical principles, where mathematical formulas provide a quantitative expression for a specific case of interest. In the case of projectile motion the key physical principle is conservation of energy, where the resulting mathematical expressions come from the application of the conservation of energy principle to a moving object with the assumption that frictional effects can be ignored. These defining equations can then be rearranged and combined to allow calculation of quantities of interest from the experimentally measured features of the trajectory like D and H.

Sample Problems

1. A pool ball leaves a 0.60-meter high table with an initial horizontal velocity of 2.4 m/s. Predict (i) the time required for the pool ball to fall to the ground and (ii) the horizontal distance between the table's edge and the ball's landing location.

   The relevant equation is

   $$y = (v_{ox} \times t) + (0.5 \times g \times t^2)$$

   That will allow us to determine the time it takes for the pool ball to hit the ground, remember that the y-component of the initial velocity v_{oy} is zero, since the initial velocity is only in the x-direction

   $$-0.6m = (0\,m/s \times t) + (0.5 \times -9.8\,m/s^2 \times t^2)$$

   $$-0.6m = (-4.9\,m/s^2)t^2$$

   $$\sqrt{\frac{-0.6m}{-4.9\,m/s^2}} = t^2$$

   $$\sqrt{1.22s^2} = t = 0.35s$$

   Once the time has been determined, the horizontal distance the pool ball travels can be determined using the formula

   $$x = v_x \times t$$

   $$x = 2.4\,m/s \times 0.35s = 8.4m$$

2. A soccer ball is kicked horizontally off a 22.0-meter high hill and lands a distance of 35.0 meters from the edge of the hill. Determine the initial horizontal velocity of the soccer ball.

   The relevant expression here is that the y-component of the initial velocity v_{oy} is zero.

   $$y = 0.5 \times g \times t^2$$

   $$\sqrt{\frac{y}{0.5 \times g}} = t$$

   $$t = \sqrt{\frac{22.0m}{0.5 \times 9.8\,m/s^2}} = 2.1s$$

Once again, when the time has been determined, the distance the pool ball travels can be determined using the formula

$$x = V_{ox} \times t$$

$$\frac{x}{t} = V_{ox}$$

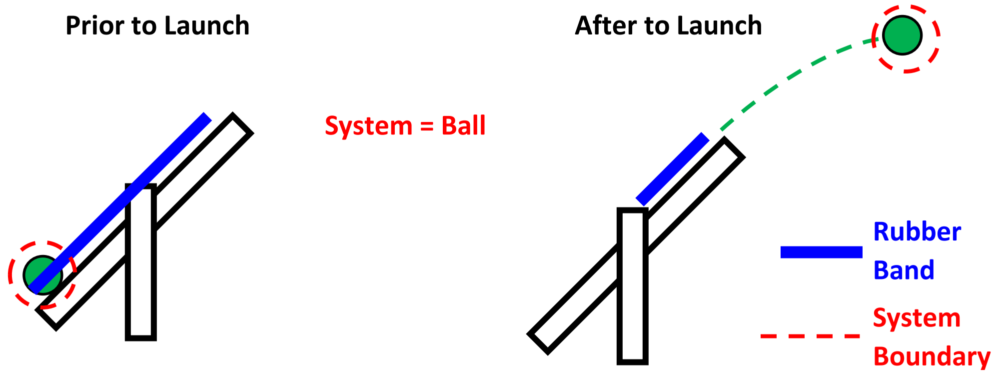
$$V_{ox} = \frac{35m}{2.1s} = 16.7\,m/s$$

Lesson 5: Mechanical Energy

Conservation of Energy is one of the key principles of physical science, where the total energy is composed of kinetic energy and potential energy. **Conservation of Energy** states that the total energy of an **isolated system** cannot change, although there can be interconversion of the various types of energy in the system. In the pendulum example in Lesson 1 and the projectile examples in Lessons 2, 3 and 4 kinetic energy was interconverted with gravitational potential energy. An isolated system is one where (i) no work is done through the system boundary and (ii) no heat flows through the system boundary.

In order to apply Conservation Energy it is important to precisely define the **system** under consideration. Consider the schematic of ball launcher experiment shown below, where there are two different ways to define the system.

1. System = ball. If the system is defined to be just the ball, the only two types of energy experienced by the ball are kinetic energy and gravitational potential energy. However, during the launch of the ball the rubber band is applying a force through the ball cradle to the projectile; thus, the ball is not an isolated system during the launch because an external force is applied through the system boundary.



2. System = ball + launcher. When the system is defined as the combination of the ball and the launcher, then the system has kinetic energy plus two types of potential energy – gravitational potential energy and stored mechanical energy. Prior to the launch phase of the process the gravitational potential energy is zero (since the height of the ball is zero), but there is stored mechanical energy in the rubber bands. During the flight stage of the process there is no stored mechanical energy in the rubber bands, because they have relaxed to their undeformed length; however, the ball now experience both kinetic energy and gravitational potential energy. There are no

Lesson 5: Objectives

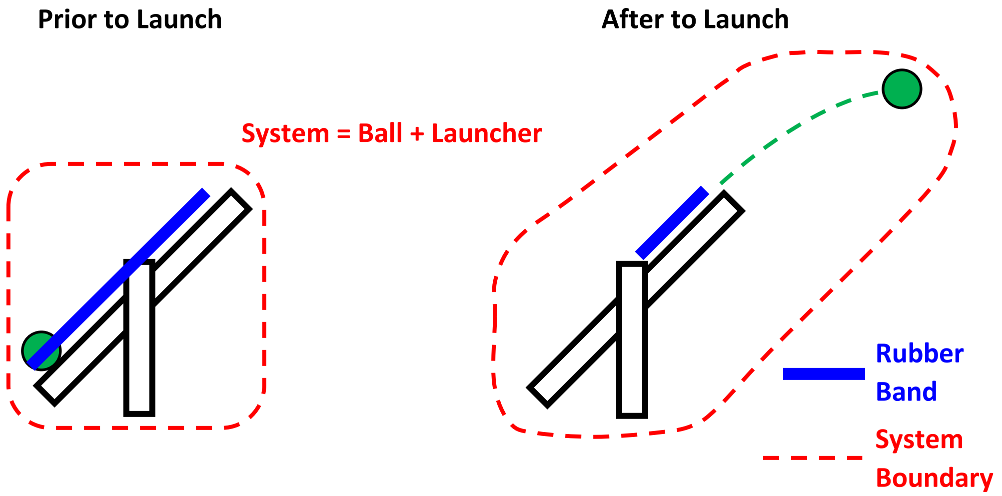
- **Apply kinematic equations to solve problems involving objects in motion.**

Content

- **Mechanical energy**
- **Isolated systems**
- **Mechanical Energy Balance**
- **Gravitational potential energy**
- **Kinetic energy**
- **Stored mechanical energy**

Sometimes the stored mechanical potential energy is called the elastic potential energy, since it is the spring-like (i.e. elastic) response of the system that stores the energy

external forces applied to either the ball or launcher, since the only force applied to the ball comes from inside the system; thus, this second alternative is an isolated system.



The ball launcher experiment can be described using either of these two approaches, but it is important to precisely state the system that is being analyzed (and not switch the system in the middle of the analysis). Determination of what is the best system to analyze a particular problem is something that takes experience, where the proper system choice often makes the analysis easier.

The sum of the kinetic energy, gravitational potential energy and stored mechanical potential energy is called the **mechanical energy**. The mechanical energy does not include stored chemical or stored electrical energy. If the system is (i) isolated and (ii) does not have any stored chemical or stored electrical energy, then the mechanical energy, ME, of the system is constant, which is equivalent to Conservation of Energy. The **Mechanical Energy Balance** given as $ME = constant$ is really just Conservation of Energy for an isolated system without stored chemical and stored electrical energy.

Mechanical Energy = Kinetic Energy + Gravitational Potential Energy + Stored Mechanical Energy

In the pendulum example, the natural choice for the system is *the oscillating mass of the pendulum*, where there is both kinetic energy and gravitational potential energy. For the pendulum there is no stored mechanical energy anywhere in the whole device. Also, there are no external forces and no heat flow; thus, it is an isolated system and $ME = constant$.

In the ping-pong ball launcher example, defining the system as the *ball plus launcher* allows one to incorporate not just the kinetic energy and gravitational potential energy of the ball during flight, but also the potential energy stored in the rubber band prior to launch. There is no external work and no heat flow through the boundary of ball plus launcher system; thus, when the system is defined as the ball plus launcher it is an isolated system. And for an isolated system without stored chemical or stored electrical energy, the Conservation of Energy principle reduces to $ME = constant$.

Ping-Pong Ball Launcher

Now reconsider the mechanical energy of the of the ping-pong ball launcher that was studied in Lesson 3. When the launcher is loaded just prior to launching the ball, the kinetic and gravitational potential energy are zero, where the stretched rubber band has stored mechanical energy that depends upon (i) the spring constant k of the rubber band and (ii) how far the rubber band has been deformed. The stored mechanical energy in the rubber band is converted into a combination of kinetic energy $KE = mv^2/2$ and gravitational potential energy $PE = mgh$ of the ball during its flight, v is the ball's speed and h is the ball's height.

A member of the Korean archery team at the 2012 London Olympics

Bow and Arrow

A bow and arrow is an excellent example of a system that has all three components of mechanical energy: kinetic energy, gravitational potential energy and stored mechanical energy. The system is defined as the bow and the arrow. The bow acts like a spring that can store mechanical energy when it is pulled back as shown in the schematic below. A typical distance L that the string on the bow is pulled back is 1 *ft*, although this distance depends upon the size of the archer. The resistance to bending the bow is given in terms bow force, which is how much force F must be applied to flex the bow just prior to launching the arrow. Bows come with different resistance that can range from 30 pounds force for smaller women to 50 pounds force for larger women and typical men and up to 70 pounds force for very strong men. For a 50 lb_f bow that is pulled back 1 *ft*, the bow's spring constant defined by $F = kL$ is $k = F/L = 50 lb_f/ft$. In Module 3: Lesson 1 the potential energy stored in a spring is $PE = kL^2/2$, which for the current example is $PE = kL^2/2 = 50 ft lb_f = 67.8 Nm$. Since there is no kinetic energy and no gravitational potential energy (the height h is defined as the distance the arrow is from the ground) prior to release of the arrow, the mechanical energy in the bow and arrow system just prior to launch is $67.8 Nm$ If we assume that 100% of the bow's stored mechanical energy is converted into the arrow's kinetic energy when it is launched at time $t = 0$; specifically,

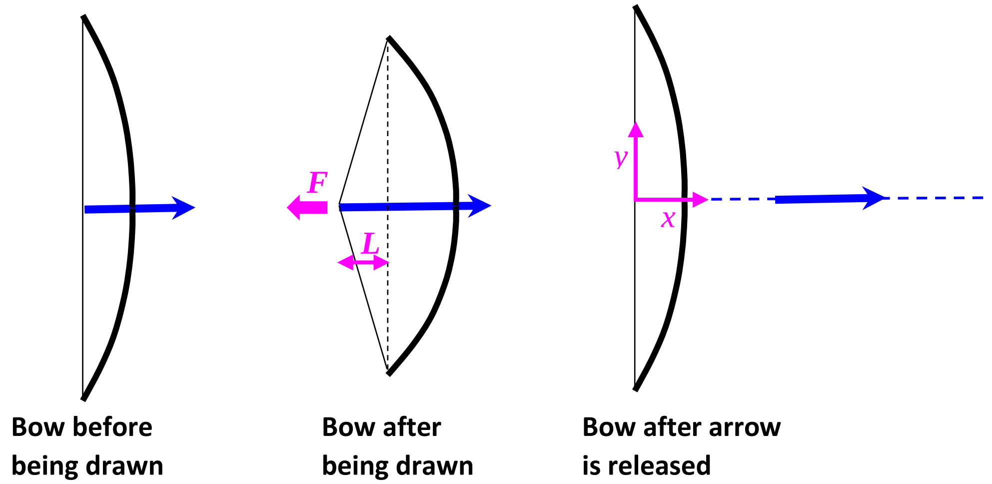
Remember lb (or more precisely a pound mass lb_m) is a measure of an object's mass, whereas the pound force lb_f is a measure of force (i.e. like Newtons in the SI system) which is force experienced by a pound mass due to gravity at the earth's surface.

$$PE_{bow} = KE_{arrow}(t = 0) = m_{arrow} \times \frac{v_o^2}{2}$$

A typical mass of a lightweight arrow use in target practice is 300 grains, where grains are the units of mass used in archery, where 100 grains = 6.48 g; thus, the mass of this particular arrow is 19.4 gm. Now solving for the magnitude v_o of the initial velocity

$$v_o = \sqrt{\frac{2 \times PE_{bow}}{m_{arrow}}} = \sqrt{\frac{2 \times 67.8 Nm}{19.4 g}} = \sqrt{\frac{2 \times 67.8 kg \cdot m/s^2}{19.4 g} \times \frac{1000 g}{kg}} = 83.6 \, m/s$$

where this answer assumes that there is no frictional resistance between the moving arrow and the air.



For an Olympic archery range the distance D to the target is 70 m. Gravity acts on the pulling it downwards during its flight, where equations describing projectile motion were developed in Lesson 4. We know the initial velocity and the distance D to the target. If the target is at the same height as the arrow when it is launched, what is the angle θ from the horizontal that must be used in order for the arrow to hit the bullseye? The equation that has the distance D, the initial velocity and the launch angle come from Eqn. 4.19a, which we used several times in Lesson 4. Specifically,

$$D = \frac{2v_{ox}v_{oy}}{g} = \frac{2v_{ox}\cos\theta\, v_{oy}\sin\theta}{g}$$

And upon rearrangement

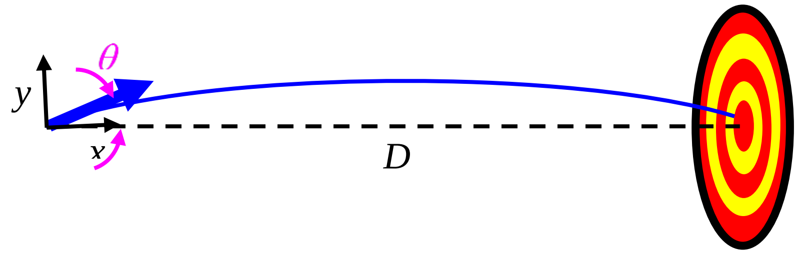
The angle *q* in trigonometric functions like $\sin\theta$ and $\cos\theta$ are naturally expressed in the units of radians a full circle (i.e. 360 degrees) is 2π radians. Thus, the conversion between radians and degrees is $180^o = \pi$ radian = 3.1416 radians

$$\cos\theta\sin\theta = \frac{Dg}{2v_o^2} = \frac{70m \times 9.8\, m/s^2}{2\times(83.6\, m/s)^2} = 0.0491$$

We now have an equation defining θ, but it is complicated. However, if the angle is small $\cos\theta\sin\theta$ is approximately equal to θ, i.e. is 0.0491 radians = 28.1^o. Thus, one needs to aim the arrow a little higher than the bullseye because gravity pulls the arrow downward. Using our sin and cos relationships, $v_{ox} = v_o\cos\theta = 83.6\, m/s \times \cos 28.1^o = 83.5\, m/s$ and $v_{oy} = v_o\sin\theta = 83.6\, m/s \times \sin 28.1^o = 4.1\, m/s$. (A table of the values of $\sin\theta$, $\cos\theta$, and $\tan\theta$, are given at the end of this lesson). Finally, the time of flight of the arrow to the target can be determined using Eqn. 4.19a, where

$$t_D = \frac{2v_{oy}}{g} = \frac{2\times 4.1\, m/s}{9.8\, m/s^2} = 0.854s$$

The time of flight of the arrow is just a little less than one second.



Fun activity

There are toy crossbows that shoot Nerf projectiles. Have the students analyze the Nerf crossbow using the ideas described above, determining how the distance that the Nerf projectile travels depending upon the launch angle and how far the crossbow is deformed.

Bowling Ball and Pins

A bowling ball typically has a speed of approximately 20 mph when released and a speed of approximately 16 mph when it hits the pins. A 'strike' occurs when all ten pins are set into motion upon being hit by the bowling ball. What is the speed of the ten bowling pins after being struck by the bowling ball, assuming that all ten pins have the same velocity v_p after being hit. The mass of a bowling ball is 5 kg and the mass of an individual bowling pin is 1.6 kg.

The <u>system</u> is defined to be the bowling ball plus the ten pins. Applying the mechanical energy balance, the only energy is the kinetic energy of the bowling ball and the kinetic energy of the pins, since (i) there is no stored mechanical energy and (ii) there is no change in the gravitational potential energy since neither the bowling ball or pins change their height. Prior to the ball striking the pins, only the bowling ball has kinetic energy, and after striking the pins both the ball and the pins have kinetic energy. Thus, the mechanical energy balance (really conservation of energy since there is no thermal, gravitational potential energy, stored mechanical, stored chemical or stored electrical energy) gives

$$\frac{1}{2}m_B v_{B,i}^2 = \frac{1}{2}m_B v_{B,f}^2 + 10 \times \frac{1}{2}m_F v_{F,i}^2$$

where m_B is the mass of the bowling ball, m_p is the mass of each of the ten bowling pins, $v_{B,i}$ is the initial velocity of bowling ball, $v_{B,f}$ is the final velocity of the bowling ball after striking the pins and $v_{p,f}$ is the final velocity of each of the bowling pins. In the mechanical energy balance equation above m_B, m_p and $v_{B,i}$ have known values, where $v_{B,f}$ and $v_{p,f}$ are unknown. A basic principle of mathematics is that if you have two unknowns, you will need two equations to solve for those unknowns. Thus, we need another piece of information in order to determine $v_{B,f}$ and $v_{p,f}$. If the final velocity of the bowling ball $v_{B,f}$ was given, that would be one additional piece of information. Or, another, independent physical principal could be that information. So far, we have used **Conservation of Energy** (i.e., the mechanical energy balance for this specific problem). In Module 8 Newton's 2nd Law will be introduced which is **Conservation of Momentum** - an additional physical principle that yields a second equation that can be used with the equation above that came from Conservation of Energy. Thus, the question posed at the beginning of this example cannot be answered without additional knowledge.

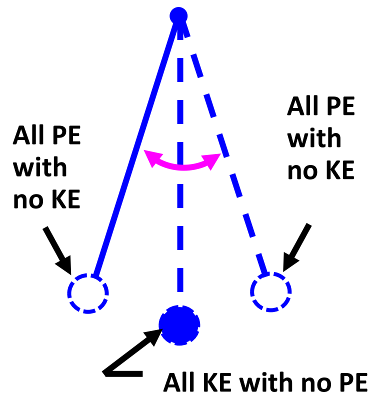
The bowling ball example will be revisited after introduction of the Conservation of Momentum principle in a future module.

More generally if one has *N* unknown quantities, then you will need N equations to solve for the *N* unknowns.

Wrecking Ball

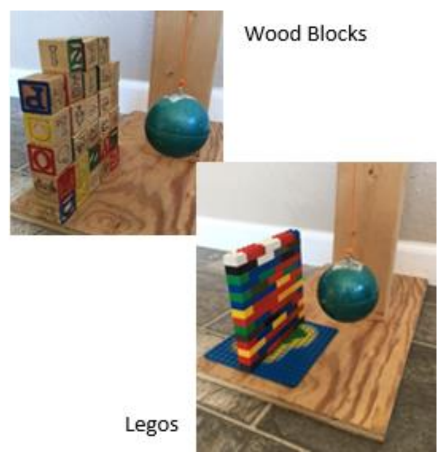
A heavy wrecking ball that swings from a tall crane is often used to demolish old buildings, were the wrecking ball swings from the crane like a pendulum. The system is defined as the wrecking ball after it has been released. All of the energy is mechanical energy. The mechanical energy is exchanged between the kinetic energy in the wrecking ball as it swings and the gravitational potential energy in the ball as its height was increased by pulling back just prior to release. In order for the ball to have the maximum mechanical energy, the ball needs to be raised as far as possible just prior to release, which is accomplished by having the crane positioned to maximize the angle that the cable holding the wrecking ball makes with the vertical just prior to release. The ball has the maximum kinetic energy (and thus the maximum wrecking capability) when the cable and wrecking ball is vertical. Thus, the tip of the crane's arm needs to be located directly over the part of the building to be struck with the ball. The mechanical energy balance is what controls the basic operations of a complex construction operation like a wrecking ball that is used to demolish buildings.

'Lady' – a crane equipped with a 4 ton wrecking ball – is used to demolish an old hospital in Norristown, PA.



Fun activity

The wood stand that was constructed for pulley (Module 2: Lesson 2) and energy storage experiment (Module 3: Lesson 2) can serve as the crane. A bolt with washers and nuts can be a 'wrecking ball', where a string that attaches the 'wrecking ball' to the wood stand is the cable. Build a wall using Legos. Start out with a small wrecking ball (i.e. only the bolt) and only pull back the ball a little – does it demolish the wall (hopefully not)? Then increase the angle and/or increase the weight of your wrecking ball by adding more washers/nuts. When does the wall break? You can do this for different types of wall construction to see what type of Lego wall is the strongest.



The physical science of a rocket will be better understood after an introduction of chemical reactions and the gas law discussed during the Chemistry Modules.

Rockets

A rocket looks like a projectile. Can it also be analyzed using the mechanical energy balance? The rocket surely has kinetic energy due to its velocity and a change in the gravitational potential energy as it takes off. However, a major feature of a rocket is how the rocket fuel is burns with exhaust gasses exiting through the rocket nozzle that cause propulsion of the rocket. The chemical energy of the rocket fuel is transformed into the kinetic energy of the exhaust gasses that in turn provide propulsion, i.e. kinetic energy, of the rocket. Moreover, the rocket is not an isolated system, since gasses are leaving the rocket. Thus, the mechanical energy balance which only is concerned with systems that have kinetic energy, stored mechanical energy and gravitational potential energy is not applicable to the rocket.

Module 7 Activities and Resources

Lesson 1: Motion Review and Kinematics PowerPoint Slide deck
Kinematics Student Activity Sheet

Lesson 2: Two-Dimensional Motion Review PowerPoint Slide deck
Characteristics of Projectile Motion Student Activity Sheet

Lesson 3: Ping-Pong Ball Launcher Investigation Data Sheet

Lesson 4: Analyzing Projectile Motion PowerPoint Slide deck
Projectile Motion Analysis Student Activity Sheet

Lesson 5: Law of Conservation Research Activity Sheet

The following is an incomplete list of referenced resources

http://www.physicsclassroom.com
https://www.mathsisfun.com/physics/energy-potential-kinetic.html
https://www.omnicalculator.com/physics/projectile-motion

The Purpose behind all resources within the hardware store science curriculum is to research the effective integration of STEM subjects into a physical science classroom. All material is organized from outside sources and solely intended to provide the researchers a framework for the development of original content based on experimental findings.

