Resource Theory of Contextualtiy

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November 2018

Resource Theory

Powerful framework for the formal treatment of a physical property as an operational resource, adequate for its characterization, quantification, and manipulation.

B. Coecke, T. Fritz, R. W. Spekkens, **A mathematical theory of resources**, Information and Computation (2016).

Resource Theory

- Objects and free objects;

Resource Theory

- Objects and free objects;
- Free operations;

Resource Theory

- Objects and free objects;
- Free operations;
- Quantifiers;

Resource Theory

- Objects and free objects;
- Free operations;
- Quantifiers;
- Relation between quantifiers and applications.

Contextuality

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Set of measurements in a physical system

$$\mathcal{X} = \{\mathrm{X}_1, \mathrm{X}_2, \ldots, \mathrm{X}_n\}$$

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Set of **contexts**

$$\mathcal{C} \subset \mathcal{P}(\mathcal{X})$$

that represent subsets of measurements in $\mathcal{X}$ that can be jointly performed.

$$\mathcal{X} = \{A, B, C\}$$

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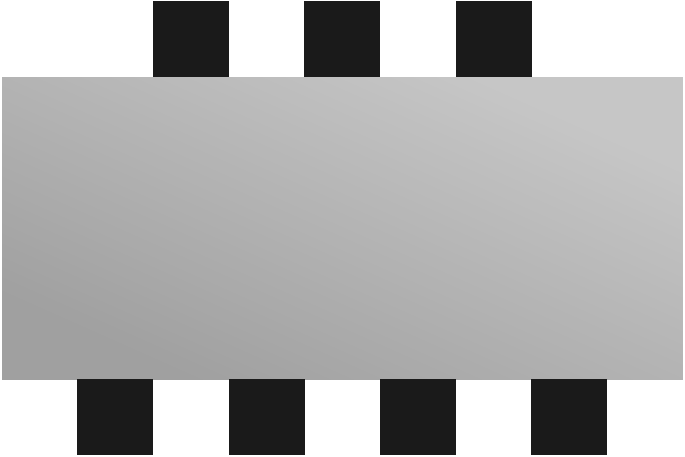
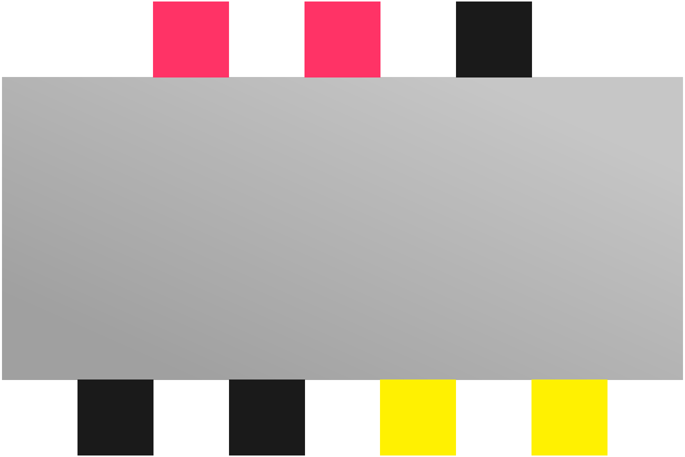
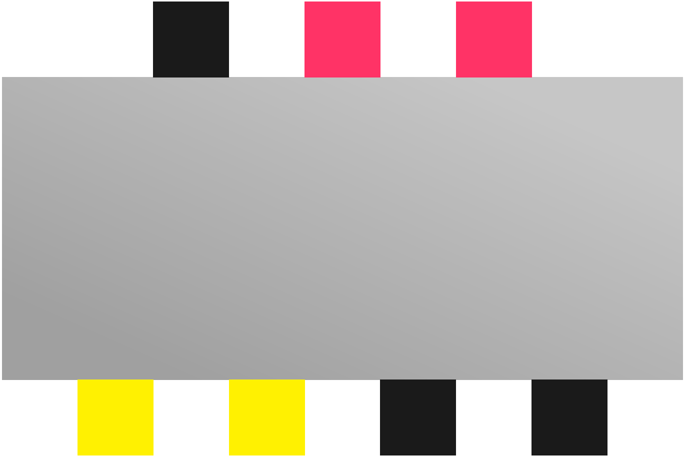
$$\mathcal{C} = \{\{A, B\}, \{B, C\}\}$$







Behavior

A set of probability distributions $p_{\mathrm{C}}(\cdot)$, one for each context C.

Behavior

$$p_{AB}(ab)$$

$$p_{BC}(bc)$$

Nondisturbing behaviors

Marginal probabilities of $p_{\mathrm{C}}(\cdot)$ and $p_{\mathrm{C}'}(\cdot)$ coincide in the intersection $\mathrm{C} \cap \mathrm{C}'$.

Non-disturbance

$$\sum_b p_{AB}(ab) = \sum_c p_{BC}(ac) = p_B(b)$$

Noncontextual (classical) behaviors

There is a global probability distribution $p_{\mathcal{X}}(\cdot)$ whose marginals coincide with $p_{\mathrm{C}}(\cdot)$ for every context C.

Noncontextual (classical) behaviors

$$p_{ABC}(abc)$$

Noncontextual (classical) behaviors

$$p_{ABC}(abc)$$

$$p_{AB}(ab) = \sum_{c} p_{ABC}(abc)$$

$$p_{BC}(bc) = \sum_{a} p_{ABC}(abc)$$

Resources for Quantum Communication and Quantum Computation

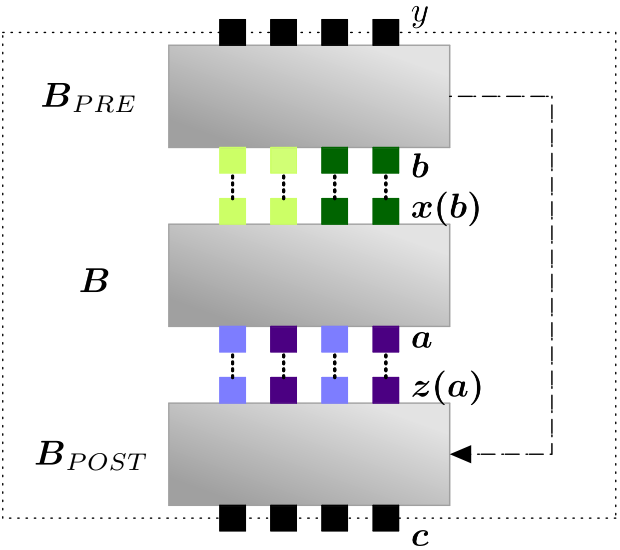
Contextuality is good: related to advantages of quantum systems over their classical counterparts.

Free operations: Noncontextual Wirings

B. Amaral, M. Terra Cunha, A. Cabello, and L. Aolita **Noncontextual Wirings** Phys. Rev. Lett. (2018).

Operations that transform one box into another, possibly in a different scenario, and such that the image of a noncontextual box is a noncontextual box.

Noncontextual wirings



Nondisturbance preservation

The nondisturbing set is preserved under the action of noncontextual wirings.

B. Amaral, A. Cabello, M. Terra Cunha, L. Aolita, **Noncontetual Wirings**, Physical Review Letters (2018).

Noncontextuality preservation

The noncontextual set is preserved under the action of noncontextual wirings.

B. Amaral, A. Cabello, M. Terra Cunha, L. Aolita, **Noncontetual Wirings**, Physical Review Letters (2018).

Quantifiers

Contextuality Quantifiers

A Contextuality Quantifier is any function

$$Q : ND \longrightarrow \mathbb{R}^{+}$$

such that

$$Q(B) = 0 \ \text{ iff } \ B \in NC$$

and

$$Q\left(\mathcal{W}\left(B\right)\right) \leq Q\left(B\right)$$

for any noncontextual wiring $\mathcal{W}$.

Relative Entropy of Contextuality

Relative Entropy

The relative entropy S_{d} of p relative to q (over the same sample space) is

$$S_{\mathrm{d}}(p \| q) = \sum_{i \in \Omega} p(i) \log \left(\frac{p(i)}{q(i)} \right).$$

Relative Entropy between Behaviours

$$S_B = \max_{C} S_{\mathrm{d}}\left(p_C \| p'_C\right)$$

Relative Entropy of Contextuality

$$R_{\mathrm{C}}(\boldsymbol{B}) = \min_{\boldsymbol{B}' \in \mathsf{NC}} S\left(\boldsymbol{B} \| \boldsymbol{B}'\right)$$

Monotonicity of R_{C}

$$R_{\mathrm{C}}\left(\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})\right) \leq R_{\mathrm{C}}\left(\boldsymbol{B}\right).$$

B. Amaral, A. Cabello, M. Terra Cunha, L. Aolita, **Noncontetual Wirings**, PRL.

$$\boldsymbol{B}_{\mathrm{f}}^{\cdot}) = \max_{\psi\in\mathcal{I}_{\mathcal{Y}}} S_{\mathrm{d}}\left(\boldsymbol{P}_{\mathcal{C}|\mathcal{Y}}(\cdot,\psi)\,\|\boldsymbol{P}'_{\mathcal{C}|\mathcal{Y}}(\cdot,\psi)\right\} \tag{33}$$

$$= \sum_{\boldsymbol{c}} p_{\mathcal{C}|\mathcal{Y}}(\boldsymbol{c},\psi^*) \log\left(\frac{p_{\mathcal{C}|\mathcal{Y}}(\boldsymbol{c},\psi^*)}{p'_{\mathcal{C}|\mathcal{Y}}(\boldsymbol{c},\psi^*)}\right) \tag{34}$$

$$= \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*) \log\left(\frac{\sum_{\boldsymbol{a},\boldsymbol{b}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*}{\sum_{\boldsymbol{a},\boldsymbol{b}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p'_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*}\right. \tag{35}$$

$$\leq \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*) \log\left(\frac{p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*)}{p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p'_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*)}\right) \tag{36}$$

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$$= \sum_{\boldsymbol{a},\boldsymbol{b}} p_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*) \log\left(\frac{p_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a},\chi^{(\boldsymbol{b})})}{p'_{\mathcal{A}|\mathcal{X}}(\boldsymbol{a},\chi^{(\boldsymbol{b})})}\right) \tag{38}$$

$$B_{\mathrm{f}}) = \max_{\psi\in\mathcal{I}_{\mathcal{Y}}} S_{\mathrm{d}}\left(\boldsymbol{P}_{\mathcal{C}|\mathcal{Y}}(\cdot,\psi)\,\|\boldsymbol{P}'_{\mathcal{C}|\mathcal{Y}}(\cdot,\psi)\right) \tag{33}$$

$$= \sum_{\boldsymbol{c}} p_{\mathcal{C}|\mathcal{Y}}(\boldsymbol{c},\psi^*)\log\left(\frac{p_{\mathcal{C}|\mathcal{Y}}(\ldots}{p'_{\mathcal{C}|\mathcal{Y}}(\ldots}\right. \tag{34}$$

$$= \sum_{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p_{\mathcal{A}|\mathcal{X}}\ldots \psi^*)\log\left(\frac{\sum_{\boldsymbol{a},\boldsymbol{b}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*}{\sum_{\boldsymbol{a},\boldsymbol{b}} p_{\mathcal{C}|\mathcal{Z},\boldsymbol{b},\psi^*}\left(\boldsymbol{c},\zeta^{(\boldsymbol{a})}\right) p'_{\mathcal{A}|\mathcal{X}}\left(\boldsymbol{a},\chi^{(\boldsymbol{b})}\right) p_{\mathcal{B}|\mathcal{Y}}(\boldsymbol{b},\psi^*}\right. \tag{35}$$

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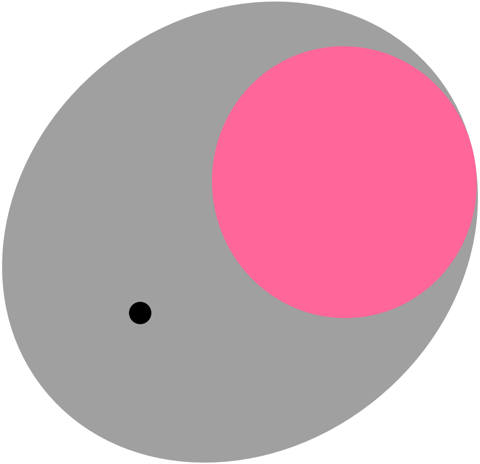
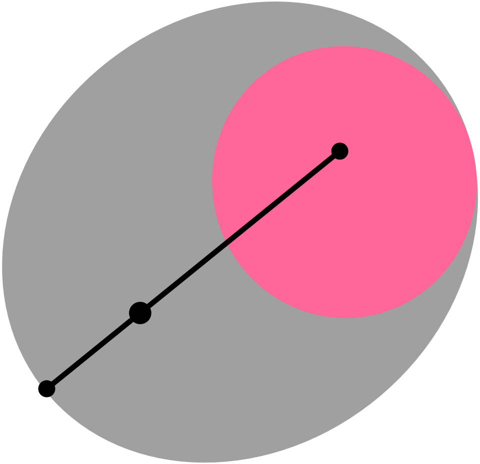
Uniform Relative Entropy of Contextuality

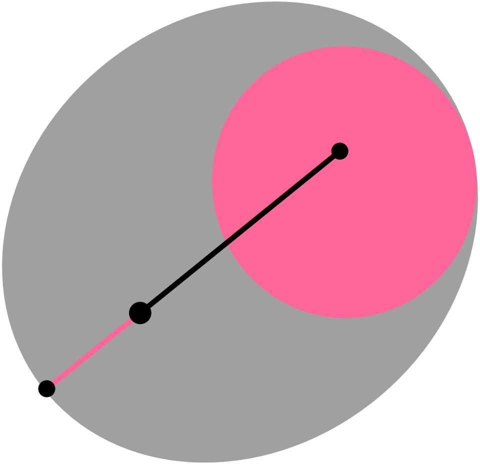
$$R_u(\boldsymbol{B}) = \min_{\boldsymbol{B}' \in \mathsf{NC}} \frac{1}{|\mathcal{C}|} \sum_{C} S_{\mathrm{d}}\left(p_C \| p'_C\right)$$

Contextual Fraction

S. Abramsky, R. S. Barbosa, S. Mansfield, **The Contextual Fraction as a Measure of Contextuality**, Physical Review Letters (2017).







Related to the succes probability for the calculation of a nonlinear Boolean function in measurement-based quantum computation.

S. Abramsky, R. S. Barbosa, S. Mansfield,**The Contextual Fraction as a Measure of Contextuality**, Physical Review Letters (2017).

Monotonicity of C

$$\mathrm{C}\left(\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})\right) \leq \mathrm{C}\left(\boldsymbol{B}\right).$$

B. Amaral, A. Cabello, M. Terra Cunha, L. Aolita, **Noncontetual Wirings**, Physical Review Letters.

Quantifiers based on the geometry of the set of nondisturbing and noncontextual behaviors

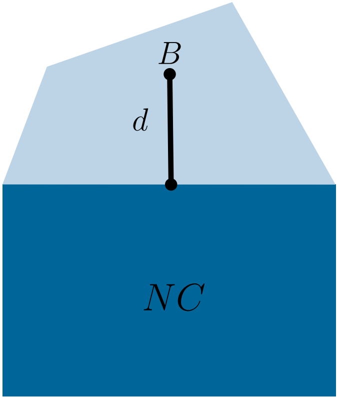
S. Brito, B. Amaral, and R. Chaves, **Quantifying nonlocality with the trace distance**, Phys. Rev. A. (2018).

B. Amaral and M. Terra Cunha, **On geometrical aspects of the graph approach to contextuality**, arXiv:1709.04812.

Geometric Quantifiers

$$D_{max}(B) = \min_{B' \in NC} \max_{C} d\left(p_C, p'_C\right)$$

B. Amaral, M. Terra Cunha, **Geometrical aspects of Contextuality**, arXiv:1710.01318.


B
d
NC

Monotonicity of D_{max}

$$\mathrm{D}_{max}\left(\mathcal{W}_{\mathsf{NC}}(\boldsymbol{B})\right) \leq \mathrm{D}_{max}\left(\boldsymbol{B}\right).$$

Geometric Quantifiers

$$D_u(B) = \min_{B' \in NC} \frac{1}{|\mathcal{C}|} d\left(B, B'\right)$$

S. Brito, B. Amaral, R. Chaves, **Quantifying Nonlocality with the Trace Distance**, PRA.

Useful not only to quantify contextuality: a powerful tool to study many other aspects.

B. Amaral, M. Terra Cunha, and A. Cabello, **Quantum theory allows for absolute maximal contextuality** Phys. Rev. A. (2015).

C. Duarte, S. Brito, B. Amaral, and R. Chaves, **Concentration phenomena in the geometry of Bell correlations**, in preparation arxiv:1810.00443.

Contextuality in general preapare-and-measure experiments

Preparation i.

Preparation i.

Measurement j with outcome k.

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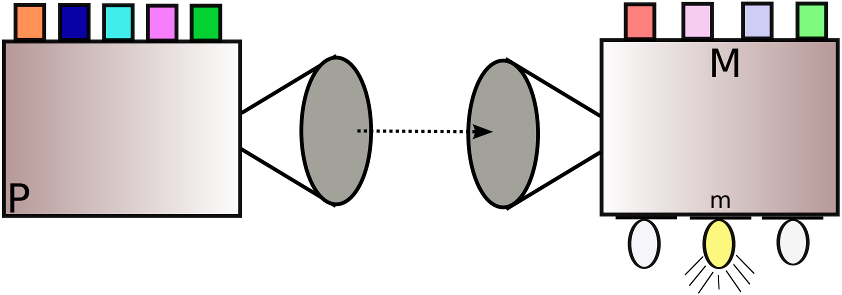
Measurement j with outcome k.

$$p\left(k|i,j\right)$$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Preparation equivalence

Two preparations i_1 and i_2 are equivalent if

$$p\left(k|i_1, j\right) = p\left(k|i_2, j\right), \quad \forall j, k$$

Measurement event equivalence

Two measurement events $k_1|j_1$ and $k_2|j_2$ are equivalent if

$$p\left(k_1|i,j_1\right)=p\left(k_2|i,j_2\right),\quad \forall i$$

Classical models

A set Λ, for each preparation i a probability distribution μ_i over Λ, and a response function $\xi_{k|j}$ such that

$$p(k|i,j) = \sum_{\lambda \in \Lambda} \mu_i(\lambda)\, \xi_{k|j}(\lambda)$$

and μ and ξ respect the operational equivalences:

$$i_1 \simeq i_2 \;\Rightarrow\; \mu_{i_1} = \mu_{i_2}$$

$$k_1|j_1 \simeq k_2|j_2 \;\Rightarrow\; \xi_{k_1|j_1} = \xi_{k_2|j_2}$$

Objects

Objects: behaviors $B = \{p\,(k|i,j)\}$.

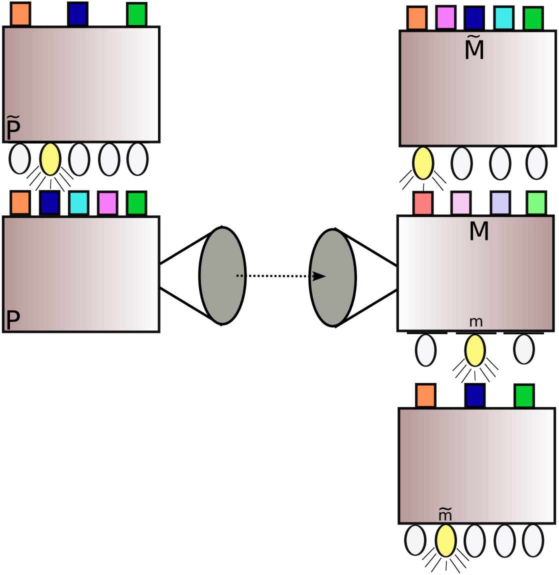
Objects

Objects: behaviors $B = \{p\,(k|i,j)\}$.

Free objects: classical behaviors.

Free operations

C. Duarte and B. Amaral, **Resource theory of contextuality for arbitrary prepare-and-measure experiments** Journal of Mathematical Physics (2018).



P̃
P
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Characterization of the set of classical behaviors in terms of linear programing. All quantifiers discussed previously can be generalized to this notion of nonclassicality as well.

D. Schimid, R. W. Spekkens, E. Wolfe, **All the nonlocality inequalities for arbitrary preapare-and-measure experiments with respect to any fixed sets of operational equivalence**, Physical Review A (2018).

Conclusions

1. Different notions of nonclassicality are necessary resources. It is important to study this feature from the perspective of resource theories.

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2. Free operations.

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4. Definition of new quantifiers based on geometric distances.

Thank you!