Hypergraph framework for Spekkens contextuality applied to Kochen-Specker scenarios

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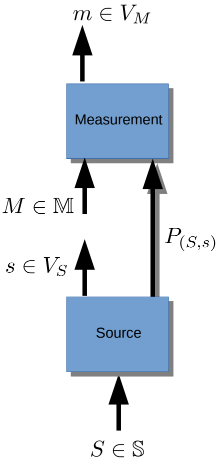
Outline

Contextuality à la Spekkens[1]

[1] R. W. Spekkens, Contextuality for preparations, transformations, and unsharp measurements, Phys. Rev. A 71, 052108 (2005).

Schematic of a prepare-and-measure scenario and its two descriptions

A prepare-and-measure scenario



Two descriptions: Operational vs. Ontological

- Operational:
$$p(m, s|M, S) \in [0, 1], \tag{1}$$
where $p(m, s|M, S) = p(m|M, S, s)p(s|S)$.

- Ontological:
$$p(m, s|M, S) = \sum_{\lambda \in \Lambda} \xi(m|M, \lambda)\mu(\lambda, s|S), \tag{2}$$
where $\mu(\lambda, s|S) = \mu(\lambda|S, s)p(s|S)$.

Features of the operational theory necessary to define noncontextuality

Operational equivalences

Preparations

- Source events:
  $[s|S] \simeq [s'|S']$, i.e.,

$$p(m, s|M, S) = p(m, s'|M, S') \quad \forall [m|M]. \tag{3}$$

- Source settings:
  $[\top|S] \simeq [\top|S']$, i.e.,

$$\sum_{s \in V_S} p(m, s|M, S) = \sum_{s' \in V_{S'}} p(m, s'|M, S') \quad \forall [m|M]. \tag{4}$$

Measurements

Measurement events are operationally equivalent ($[m|M] \simeq [m'|M']$) if no source event can distinguish them, i.e.,

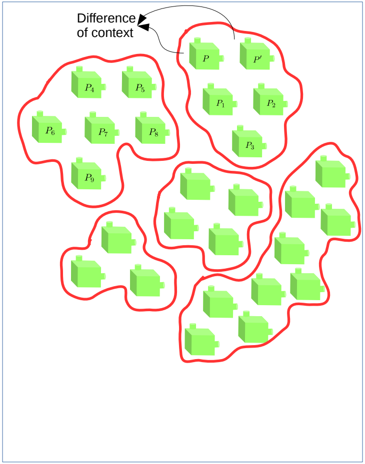
$$\forall [s|S] : p(m, s|M, S) = p(m', s|M', S), \tag{5}$$

e.g., when the same projector appears in two different measurement bases.

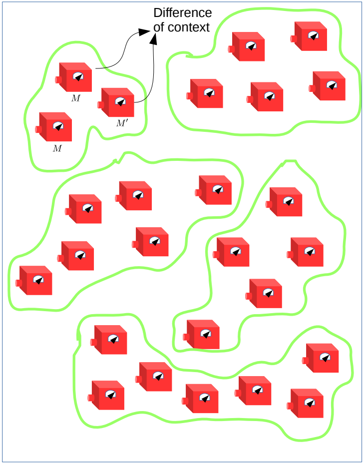
What is a ‘context’?

Any distinction between operationally equivalent procedures.

Preparations



Measurements



Examples

Preparation contexts: Different realizations of a given quantum state, e.g., different convex decompositions,

$$\frac{I}{2} = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|,$$

or different purifications,

$$\rho_A = \mathrm{Tr}_B|\psi\rangle\langle\psi|_{AB} = \mathrm{Tr}_C|\phi\rangle\langle\phi|_{AC}, \text{etc.}$$

Measurement contexts: Different realizations of a given POVM or a POVM element, e.g., same projector appearing in different measurement bases, joint measurability contexts for a given POVM, or even different ways of implementing a fair coin flip measurement.[2]

[2] Mazurek et. al., Nature Communications 7:11780 (2016).

Noncontextuality

Noncontextuality: identity of indiscernibles

If there exists no operational way to distinguish two things, then they must be physically identical.[3]

- Measurement noncontextuality:

$$[m|M] \simeq [m'|M'] \Rightarrow \xi(m|M,\lambda) = \xi(m'|M',\lambda) \quad \forall\lambda \in \Lambda$$

- Preparation noncontextuality:

$$[s|S] \simeq [s'|S'] \Rightarrow \mu(\lambda, s|S) = \mu(\lambda, s'|S') \quad \forall\lambda \in \Lambda,$$

$$[\top|S] \simeq [\top|S'] \Rightarrow \mu(\lambda|S) = \mu(\lambda|S') \quad \forall\lambda \in \Lambda.$$

[3] Equivalently: if two things are non-identical, or physically distinct, then there must exist an operational way to distinguish them.

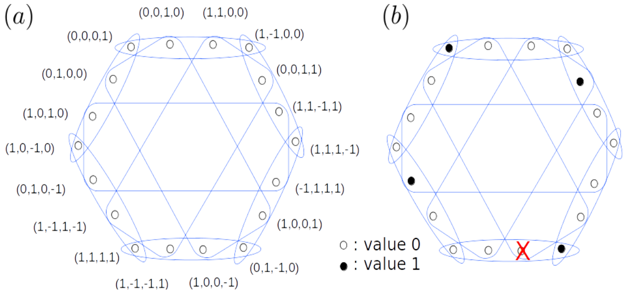
Kochen-Specker (KS) noncontextuality

KS-noncontextuality

$\Leftrightarrow$ Measurement noncontextuality and Outcome determinism [4]

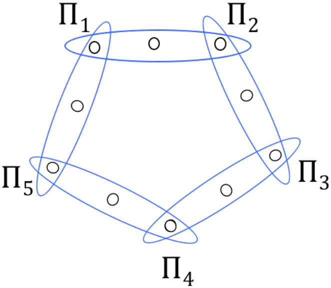
[4] Applied to measurement contexts of the type arising from joint measurability. Outcome determinism: for any $[m|M]$, $\xi(m|M,\lambda) \in \{0,1\} \quad \forall \lambda \in \Lambda$.

Kochen-Specker theorem: logical proof



Cabello et al., Physics Letters A 212, 183 (1996)

Kochen-Specker theorem: statistical proof



Klyachko et al., Phys. Rev. Lett. 101, 020403 (2008)

Kochen-Specker contextuality à la CSW [5]

[5] Cabello et al., PRL 112, 040401 (2014).

Contextuality scenario, Γ

A hypergraph Γ where the nodes of the hypergraph $v \in V(\Gamma)$ denote measurement outcomes and hyperedges denote measurements $e \in E(\Gamma) \subseteq 2^{V(\Gamma)}$ such that $\bigcup_{e \in E(\Gamma)} = V(\Gamma)$.[6]

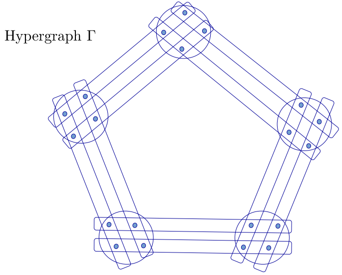


Figure: Γ for KCBS. [7]

[6] We will further assume that no hyperedge is a strict subset of another in Γ, following Acín et al (AFLS), Comm. Math. Phys. 334(2), 533-628 (2015)

[7] Klyachko et al., Phys. Rev. Lett. 101, 020403 (2008).

Orthogonality graph of Γ, i.e., $O(\Gamma)$

Vertices of $O(\Gamma)$ are given by $V(O(\Gamma)) \equiv V(\Gamma)$, and the edges of $O(\Gamma)$ are given by

$$E(O(\Gamma)) \equiv \{\{v, v'\} | v, v' \in e \text{ for some } e \in E(\Gamma)\}.$$

Probabilistic models on Γ

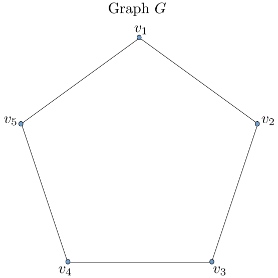
A *probabilistic model* on Γ is given by $p : V(\Gamma) \to [0,1]$ such that $\sum_{v \in e} p(v) = 1$ for all $e \in E(\Gamma)$. The set of all probabilistic models on Γ is denoted $\mathcal{G}(\Gamma)$. Relevant subsets of $\mathcal{G}(\Gamma)$:

- KS-noncontextual, $\mathcal{C}(\Gamma)$: a convex mixture of $p : V(\Gamma) \to \{0,1\}$, $\sum_{v \in e} p(v) = 1\ \forall e \in E(\Gamma)$.
- Consistently exclusive, $\mathcal{CE}^1(\Gamma)$: $p : V(\Gamma) \to [0,1]$, such that $\sum_{v \in c} p(v) \leq 1$ for all cliques c in $O(\Gamma)$.

Clearly,

$$\mathcal{C}(\Gamma) \subseteq \mathcal{CE}^1(\Gamma) \subseteq \mathcal{G}(\Gamma).$$

Exclusivity graph, G: a subgraph of $O(\Gamma)$



$$R([s|S]) \equiv \sum_{v \in V(G)} w_v p(v|S, s), \tag{6}$$

where $w_v > 0$ for all $v \in V(G)$ and $p(v|S, s)$ is a probabilistic model induced by source event $[s|S]$ on measurements events in Γ.

CSW bounds

$$
\begin{aligned}
R([s|S]) &\equiv \sum_{v \in V(G)} w_v p(v|S,s) \\
&\overset{\mathrm{KS}}{\leq} \alpha(G,w) \\
&\overset{\mathrm{Q}}{\leq} \vartheta(G,w) \\
&\overset{\mathrm{E}^1}{\leq} \alpha^*(G,w),
\end{aligned}
$$

KCBS [8] : $w_v = 1$ for all $v \in V(G)$,
$\alpha = 2$, $\vartheta = \sqrt{5}$, and $\alpha^* = 5/2$.

[8] Klyachko et al., Phys. Rev. Lett. 101, 020403 (2008).

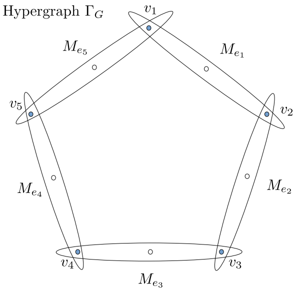
Missing ingredients?

- Measurement noncontextuality alone yields a trivial upper bound $\alpha^*(G, w)$. (Remember: no outcome determinism.)
- Need to invoke preparation noncontextuality.
- We do this next.

Hypergraph-theoretic ingredients

The contextuality scenario Γ_G

Turn maximal cliques in G into hyperedges and add an extra ("nondetection") vertex to each hyperedge.



We can now take $p(v|S, s)$ to be a probabilistic model on Γ_G rather than the full scenario Γ and retain the same probabilities on G.

Weighted max-predictability, $\beta(\Gamma_G, q)$

$$\beta(\Gamma_G, q) \equiv \max_{p \in \mathcal{G}(\Gamma_G)|_{\text{ind}}} \sum_{e \in E(\Gamma_G)} q_e \zeta(M_e, p), \tag{7}$$

where $q_e \geq 0$ for all $e \in E(\Gamma_G)$, $\sum_{e \in E(\Gamma_G)} q_e = 1$, and

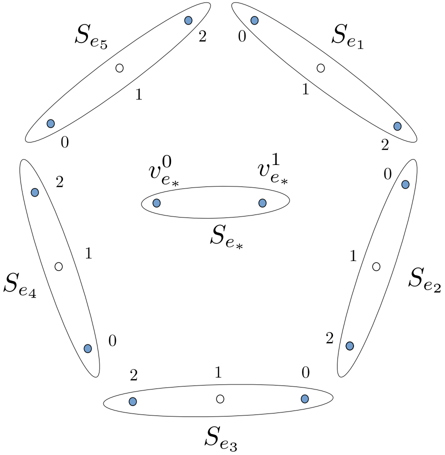
$$\zeta(M_e, p) \equiv \max_{v \in e} p(v) \tag{8}$$

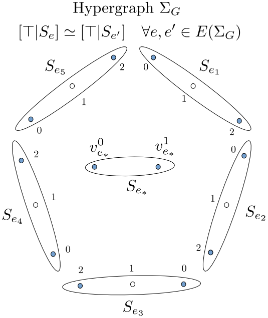
is the maximum probability assigned to a vertex in $e \in E(\Gamma_G)$ by an indeterministic probabilistic model $p \in \mathcal{G}(\Gamma_G)$.

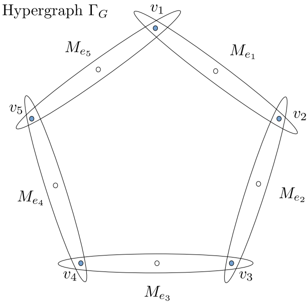
Source hypergraph

Hypergraph Σ_G

$$[\top|S_e] \simeq [\top|S_{e'}] \quad \forall e, e' \in E(\Sigma_G)$$




Hypergraph Σ_G
$[\top|S_e] \simeq [\top|S_{e'}] \quad \forall e, e' \in E(\Sigma_G)$
S_{e_1}
S_{e_2}
S_{e_3}
S_{e_4}
S_{e_5}
S_{e_*}
$v_{e_*}^0$
$v_{e_*}^1$
0
1
2


Hypergraph Γ_G
v_1
v_2
v_3
v_4
v_5
M_{e_1}
M_{e_2}
M_{e_3}
M_{e_4}
M_{e_5}

Source-measurement correlations: Corr

$$\mathrm{Corr} \equiv \sum_{e \in E(\Gamma_G)} q_e \sum_{m_e, s_e} \delta_{m_e, s_e} p(m_e, s_e | M_e, S_e), \tag{9}$$

where $\{q_e\}_{e \in E(\Gamma_G)}$ is a probability distribution, i.e., $q_e \geq 0$ for all $e \in E(\Gamma_G)$ and $\sum_{e \in E(\Gamma_G)} q_e = 1$.[9]

[9] Such that $\beta(\Gamma_G, q) < 1$ holds.

Beyond CSW:
Hypergraph framework for Spekkens contextuality

General form of the noise-robust noncontextuality inequality: KS-colourable case [10,11]

$$R([s_{e_*}=0|S_{e_*}]) \overset{\mathrm{NC}}{\leq} \alpha(G,w) + \frac{\alpha_*(G,w)-\alpha(G,w)}{p_*}\frac{1-\mathrm{Corr}}{1-\beta(\Gamma_G,q)}.$$

Here, $p_* \equiv p(s_{e_*}=0|S_{e_*}) = p(v^0_{e_*})$ and all the measurement events in G are evaluated on the source event $[s_{e_*}=0|S_{e_*}]$ to compute $R([s_{e_*}=0|S_{e_*}])$.

For the KCBS scenario: $\alpha(G,w)=2$, $\alpha_*(G,w)=5/2$, and $\beta(\Gamma_G,q)=1/2$. We then have

$$R \leq 2 + \frac{1-\mathrm{Corr}}{p_*}$$

[10] R. Kunjwal, arXiv:1709.01098 [quant-ph] (2017).

[11] R. Kunjwal and R. W. Spekkens, Phys. Rev. A 97, 052110 (2018).

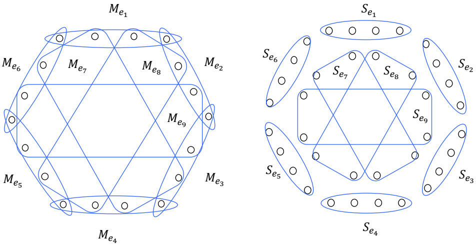
Scope of this generalization of CSW

The framework presented so far applies to KS-colourable contextuality scenarios where statistical proofs of the KS theorem apply. In particular, it covers contextuality scenarios Γ (hence also Γ_G) such that

- $\mathcal{C}(\Gamma) \neq \varnothing$,
- $\mathcal{CE}^1(\Gamma) = \mathcal{G}(\Gamma)$.

Hypergraph framework for KS-uncolourable scenarios

- For Γ such that $\mathcal{C}(\Gamma) = \varnothing$, we obtain a framework (cf. arXiv:1805.02083) based entirely on the hypergraph invariant $\beta(\Gamma_G, q)$.
- It's basic ingredients are still the contextuality scenario Γ and the corresponding source events hypergraph.



Recall

$$\beta(\Gamma_G, q) \equiv \max_{p \in \mathcal{G}(\Gamma_G)|_{\text{ind}}} \sum_{e \in E(\Gamma_G)} q_e \zeta(M_e, p), \tag{10}$$

where $q_e \geq 0$ for all $e \in E(\Gamma_G)$, $\sum_{e \in E(\Gamma_G)} q_e = 1$, and

$$\zeta(M_e, p) \equiv \max_{v \in e} p(v) \tag{11}$$

is the maximum probability assigned to a vertex in $e \in E(\Gamma_G)$ by an indeterministic probabilistic model $p \in \mathcal{G}(\Gamma_G)$.

Recall

$$\mathrm{Corr} \equiv \sum_{e \in E(\Gamma)} q_e \sum_{m_e, s_e} \delta_{m_e, s_e} p(m_e, s_e | M_e, S_e), \tag{12}$$

where $\{q_e\}_{e \in E(\Gamma)}$ is a probability distribution, i.e., $q_e \geq 0$ for all $e \in E(\Gamma_G)$ and $\sum_{e \in E(\Gamma)} q_e = 1$.[12]

[12] Such that $\beta(\Gamma, q) < 1$ holds.

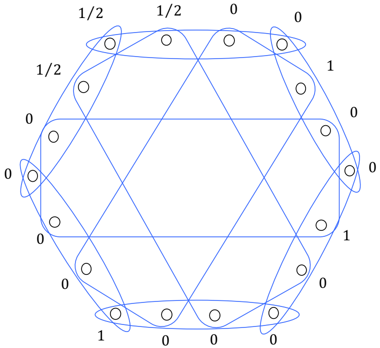
General form of the noise-robust noncontextuality inequality: KS-uncolourable case

$$\mathrm{Corr} \leq \beta(\Gamma, q). \tag{13}$$

Example: 18 ray

$$\mathrm{Corr} \leq \frac{5}{6}, \tag{14}$$

where $q_{e_i} = \frac{1}{9}$ for all $i \in [9]$.



Properties of $\beta(\Gamma, q)$ from structure of the KS-uncolourable hypergraph

- See arXiv:1805.02083 for a study of $\beta(\Gamma, q)$ for various KS-uncolourable hypergraphs.
- It presents a framework for identifying subsets of contexts (i.e., the supports of $\{q_e\}_{e \in E(\Gamma)}$) which admit a nontrivial bound on Corr given by $\beta(\Gamma, q)$.
- It applies the framework to a family of KS-uncolourable hypergraphs: those where each vertex appears in two hyperedges.

Comparision of KS vs. Spekkens

	Traditional Bell-KS approaches	Spekkens' approach
Type of context	1) ONB contexts 2) Compatibility contexts	Includes more types of contexts, for both preps and mmts.
Assumptions	MNC and OD (or at least Factorizability)	MNC and PNC (and resp. convex mixtures etc.)
Quantity of interest	Mmt-mmt correlations for a fixed input state	Also includes source-mmt correlations
Type of inequalities	Constraints on mmt-mmt corr from the classical marginal problem	More refined approach: tradeoff b/w mmt-mmt corr and source-mmt corr
KS-uncolourability proofs	Logical contradiction, no ineqs on mmt-mmt corr needed.	Robust inequality bounding source-mmt corr. No mmt-mmt corr needed.

Takeaway

1. We have obtained two complementary hypergraph-based frameworks for KS-colourable and KS-uncolourable scenarios.
2. Together, they complete the project of turning KS-type proofs of contextuality into noise-robust noncontextuality inequalities applicable to noisy measurements and preparations.
3. Open questions:
   - applications of these frameworks to quantum information?
   - hypergraph-theoretic properties of $\beta(\Gamma, q)$ vis-à-vis the structure of Γ, possible relevance to information theory?